

CHARACTERIZATION OF EQUALITY IN ZHONG-YANG TYPE (SHARP) SPECTRAL GAP ESTIMATES FOR METRIC MEASURE SPACES

SAJJAD LAKZIAN [†]

ABSTRACT. We prove that a compact $RCD^*(0, N)$ (or equivalently $RCD(0, N)$) metric measure space, (X, d, m) , with $\text{diam } X \leq d$ and its first (nonzero) eigenvalue of the Laplacian (in the sense of Ambrosio-Gigli-Savaré), $\lambda_1 = \frac{\pi^2}{d^2}$, has to be a circle or a line segment with diameter, π . This completely characterizes the equality in Zhong-Yang type sharp spectral gap estimates in the metric measure setting with Riemannian lower Ricci bounds. Among such spaces, are the familiar Riemannian manifolds with $\text{Ric} \geq 0$, $(0, N)$ –Bakry-Émery manifolds, $(0, n)$ –Ricci limit spaces and non-negatively curved Alexandrov spaces. Inspired by Gigli’s proof of the non-smooth splitting theorem, the key idea in the proof of our result, is to show that the underlying metric measure space (perhaps minus a closed subset of co-dimension, 1) splits off an interval isometrically whenever there exists a weakly harmonic potential f , the gradient flow trajectories of which, are geodesics (i.e. multiples of f are Kantorovich potentials at least for a short time and on suitable domains). This is standard in Riemannian geometry due to the de Rham’s decomposition theorem which is a key ingredient in the proof of celebrated the Cheeger-Gromoll’s splitting theorem.

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1. INTRODUCTION

Studying eigenvalues of Laplacian in different settings (for Euclidean domains, closed manifolds, manifolds with boundary, etc.) and under different geometric constraints and/or boundary conditions, is among the most intriguing and well investigated topics in Mathematics. One can construct

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iso-spectral (Riemannian) spaces that are not isometric but nevertheless, estimates on the eigenvalues provide us with plenty of geometric and analytic data about the underlying space. Estimates on the first spectral gap are particularly important because they bear valuable information about ,among other things, curvature bounds on the underlying space (geometric information) and the rate of convergence of diffusion processes to their equilibrium states (analytic information).

It is out of the scope of this article to mention all the valuable and influential research articles written in this subject over the years. We will only try to highlight the most important ones that are directly related to the problem considered in this paper namely, those dealing with (sharp) spectral gap estimates under lower bounds on the Ricci curvature and upper bounds on the diameter. In particular, we are interested in *sharp* spectral gap estimates under $\text{Ric} \geq 0$ (non-negative Ricci curvature in different settings) and $\text{diam} \leq d$. These are what we will be referring to as Zhong-Yang type spectral gap estimates. More specifically, we are concerned with the characterization of the equality case in Zhong-Yang type sharp spectral gap estimates.

Lichnerowicz [41] showed that in compact manifolds, $\text{Ric} \geq K > 0$ implies $\lambda_1 \geq \frac{nK}{n-1}$. Later, Obata [46] proved that $\lambda_1 = \frac{nK}{n-1}$ can only happen if the underlying manifold is the n - dimensional spherical space form, \mathbb{S}_K^n . This is known as the Obata's rigidity theorem.

Li [39] and Li-Yau [40], using their famous gradient estimates, proved that in a compact manifold with non-negative Ricci curvature and $\text{diam} \leq d$, one has $\lambda_1 \geq \frac{\pi^2}{2d^2}$. This estimate obviously is not sharp (for example, for a circle we know that $\lambda_1 = \frac{\pi^2}{d^2}$). Zhong-Yang [64] improved upon the work of Li and Yau and proved that one indeed has $\lambda_1 \geq \frac{\pi^2}{d^2}$ which turns out to be a sharp estimate. In 90's, Kröger [36], Bakry-Qian [9] and Chen-Wang [20, 21] unified the spectral gap results in the form of a spectral comparison with 1-dimensional model spaces. It is worth mentioning that the most general results among these, are the ones presented in Bakry-Qian [9].

The collected work of Kröger [36], Bakry-Qian [9] and Chen-Wang [20, 21] shows that for a compact Riemannian manifold M^n with $\text{Ric} \geq K$ and $\text{diam } M \leq d$ without boundary (or with convex boundary), the first non-zero (Neumann) eigenvalue, λ_1 satisfies

$$\lambda_1 \geq \hat{\lambda}(K, n, d), \quad (1)$$

where, $\hat{\lambda}(K, n, d)$ is the first (non-zero) Neumann eigenvalue of the following 1-dimensional model problem.

$$v''(x) - T(x)v' = -\lambda v \quad \text{on} \quad \left(-\frac{d}{2}, \frac{d}{2}\right) \quad \text{and} \quad v'\left(-\frac{d}{2}\right) = v'\left(\frac{d}{2}\right) = 0, \quad (2)$$

in which, the *drift term*, $T(x)$ is given by

$$T(x) := \begin{cases} \sqrt{(N-1)K} \tan\left(\sqrt{\frac{K}{N-1}}t\right) & \text{if } K > 0, 1 < N < \infty, \\ -\sqrt{-(N-1)K} \tanh\left(-\sqrt{\frac{K}{N-1}}t\right) & \text{if } K < 0, 1 < N < \infty, \\ 0 & \text{if } K = 0, 1 < N < \infty, \\ Kt & \text{if } N = \infty. \end{cases} \quad (3)$$

This comparison result has been since generalized to the n - dimensional Bakry-Émery manifolds (without boundary or with a convex one) with $\text{Ric}_N \geq K$, $n \leq N$ (see Bakry-Qian [9] and Andrews-Clutterbuck [6] and Milman [43]), ($N = n$) n - dimensional Alexandrov space without boundary (see Qian-Zhang-Zhu [48]) and n -dimensional Finsler manifolds (without boundary or with a convex one) with weighted Ricci bound $\text{Ric}_N \geq K$, $n \leq N$ (see Wang-Xia [62]). We point out that, the techniques and estimates obtained in Andrews-Clutterbuck [6] are , in nature, much

stronger and had been used in Andrews-Clutterbuck [5] to prove the fundamental gap conjecture (second spectral gap for Dirichlet eigenvalues) for Schrödinger operators in Euclidean domains.

In the context of metric measure spaces satisfying reduced weak Riemannian Ricci curvature bounds (a.k.a. $RCD^*(K, N)$ curvature-dimension conditions), the above comparison estimates have been recently proven by Ketterer [34] and Jiang-Zhang [32] and Cavalletti-Mondino [12, 13].

The most general comparison result in the setting of metric measure spaces, so far, is the above comparison estimates proven by Cavalletti-Mondino [12, 13] for p -spectral gap, $\lambda_{(X,d,m)}^{1,p}$ in essentially non-branching $CD^*(K, N)$ spaces (this class of spaces is strictly larger than the class of $RCD^*(K, N)$ spaces; for instance, it includes non-Riemannian Finsler manifolds with weighted Ricci curvature bounded below which we know are not $RCD^*(K, N)$ spaces). When, $p = 2$, one recovers the comparison results for the Laplacian. Recall that when the underlying space is $RCD(K, N)$, the p -spectral gap, $\lambda_{(X,d,m)}^{1,p}$ coincides with the first non-zero eigenvalue of the p -Laplacian. Their proof cleverly uses $1 - D$ localization of curvature-dimension conditions in metric measure spaces (developed earlier by Cavalletti [11]) and hence is different in nature from the other results.

Obata's rigidity theorem is the first result that characterizes the equality case of these spectral gap estimates (when $K > 0$). In general (i.e. when the underlying space is not necessarily smooth), for $K > 0$, the rigidity result states that $\lambda_1 = \hat{\lambda}(K, n, d)$ if and only if the underlying space is a *spherical suspension* (see Ketterer [34], Jiang-Zhang [32] and Cavalletti-Mondino [13]).

For compact Riemannian manifolds with $\text{Ric} \geq 0$ ($K = 0$), Hang-Wang [31] characterized the equality case in Zhong-Yang's spectral gap estimates. They proved that $\lambda_1 = \frac{\pi^2}{d^2}$ if and only if M is a circle of perimeter 2π (if M is without boundary) or a line segment of length π (if M has a convex boundary).

In Finsler structures with non-negative weighted Ricci curvature, $\text{Ric}_N \geq 0$, such a rigidity result has been recently proven by Xia [63].

The spectral comparison for p -Laplacian in place of the usual Laplacian and the equality case has been considered, among others, by Valtorta [60] and Naber-Valtorta [45].

In this paper, we will attend to the equality case of Zhong-Yang type sharp spectral gap estimates in the general setting of metric measure spaces satisfying $RCD^*(0, N)$ (or equivalently $RCD(0, N)$) curvature-dimension conditions. These spaces are reminiscent of Riemannian manifolds with non-negative Ricci curvature and dimension less than or equal to N . The main theorem of this article is the following.

Theorem 1.1. *Suppose (X, d, m) is a compact $RCD(0, N)$ space with $\text{diam } X \leq d$ and $\text{supp}(m) = X$. If $\lambda_1 = \frac{\pi^2}{d^2}$, then, X is either a circle or a line segment.*

The proof, roughly speaking, follows the same ideas as in the proof in the smooth setting (see Hang-Wang [31]) though, since we are only allowed to use the calculus tools currently available in $RCD(K, N)$ spaces, we inevitably have to adapt many arguments to the setting of metric measure spaces (which all can be expressed solely in terms of the metric, d , and the measure, m .)

To provide a road map, in below, we highlight the main steps of the proof:

- Step I :** By rescaling the metric, d , we will assume that $\text{diam}(X) = \pi$ and $\lambda_1 = 1$. Then, we pick an eigenfunction, u associated to $\lambda_1 = 1$. Using the *self-improvement property* of Bakry-Émery curvature-dimension conditions (as in Savaré [52] or Sturm [56]), *Bochner inequality* and the *strong maximum principle* in non-smooth setting (see Björn-Björn [10]), we prove that $u^2 + |\nabla u|^2 \equiv 1$. Then, we show that $f(x) := \sin^{-1}(u(x))$ is weakly harmonic and $|\nabla f| \equiv 1$.
- Step II :** We show that for any $a \in (-\frac{\pi}{2}, \frac{\pi}{2})$, (af) is $\frac{d^2}{2}$ -concave on a maximal domain, \mathcal{D}_a (which is a geodesically convex subset of X). We will then prove the short time existence of the gradient flow of f and the fact that the *gradient flow trajectories* are indeed geodesics.
- Step III :** With arguments that are similar in essence to Gigli's proof of the non-smooth splitting theorem (see Gigli [25, 26]), we will prove that the *harmonicity* of f will result in the gradient flow of f (whenever defined) being measure preserving and roughly speaking, f being "Hessian free" will imply that the gradient flow of f is distance preserving. Then, defining a suitable quotient space (X', d', m') (which *a posteriori* will be $= f^{-1}(\{0\})$) and letting $S_{\pm 1} := u^{-1}(\{\pm 1\}) = f^{-1}(\{\pm \frac{\pi}{2}\})$, we will demonstrate that $\hat{X} := X \setminus (S_{-1} \sqcup S_{+1})$ splits isometrically as

$$(X \setminus (S_{-1} \sqcup S_{+1}), d, m) \stackrel{\text{isom}}{\cong} \left(\left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \times X', d_{\text{Euc}} \oplus d', \mathcal{L}^1|_{(-\frac{\pi}{2}, \frac{\pi}{2})} \otimes m' \right),$$

and that X' in fact, satisfies the $RCD(0, N-1)$ curvature-dimension conditions.

- Step IV :** Finally, using the rather elementary *Pythagorean theorem* for the underlying space that is now in the form of an isometric product space, we will argue that connected components of X' have to be 0-dimensional. This means that X is 1-dimensional and hence it has to be a circle or a line segment. This last conclusion can be proven directly using the fact that $RCD(K, N)$ are essentially non-branching or by using similar ideas to those employed in Kitabepu and the author's characterization of low dimensional $RCD^*(K, N)$ spaces (see Kitabepu-Lakzian [35]).

Remark 1.1. *A posteriori, it will become clear that in fact, the whole space, X , splits isometrically, but in order to avoid technical difficulties when the gradient flow trajectories of f reach their end points (and we know that the gradient flow Ψ_t will lose its additive property and consequently measure and distance preserving properties when "t" is large), we need to first take out the extremal sets $S_{\pm 1}$ and then prove the splitting. But at the end, this will not be detrimental to the proof at all. In general, we need to keep in mind that unlike the splitting theorem (where we have a line in a complete underlying space and that the gradient flow of the Busemann function comes from the action of a group), in here, the gradient flow on \hat{X} is induced by a groupoid action.*

The following is an obvious corollary of Theorem 1.1.

Corollary 1.2. *Let (X, d, m) be a compact metric measure space with $\text{diam } X = \pi$. Let*

$$N := \inf \{ N' : X \text{ satisfies } RCD(0, N') \text{ curvature-dimension conditions} \}.$$

Then if $N \geq 2$, we have $\lambda_1 > 1$ or else, X is a weighted closed interval.

This can also be stated locally in terms of the convexity radius of a point.

Corollary 1.3. *Let (X, d, m) be a complete and locally compact $RCD(0, N)$ metric measure space. Suppose, for some $r > 0$, $B_r(x)$ is geodesically convex, and the Hausdorff dimension, $\mathcal{H}_{\text{dim}}(B_r(x)) > 1$ then,*

$$\lambda_1(\Delta_{ch}^{B_r(x)}) > \frac{\pi^2}{\text{diam}(B_r(x))^2} \geq \frac{\pi^2}{4r^2}.$$

Another direct consequence of Theorem 1.1 is the following estimate on the rate of convergence of diffusion processes generated by Δ_{Ch} to their harmonic states.

Corollary 1.4. *Let (X, d, m) be a compact $RCD(0, N)$ metric measure space (as above, " N " is assumed to be the smallest among such dimensions). If the Hausdorff dimension $\mathcal{H}_{\text{dim}}(X) \geq 2$ then, the rate of the convergence to equilibrium of the diffusion process generated by Δ_{Ch} in X is strictly faster than the rate of the convergence to equilibrium of the diffusion process generated by the operator $\frac{d^2}{dx^2}$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.*

Remark 1.2. *It is not hard to see that similar arguments (as are made in this article) can also be adapted to p -Laplacian instead of Δ_{Ch} . In this setting, after finding the proper PDI, one can apply a maximum principle for p -sub-minimizers (for some discussions regarding p -sub-minimizers, see Gigli-Mondino [27]). In our upcoming work (Lakzian [38]), we prove sharp spectral gap estimates for p -Laplacian in $RCD(K, N)$ spaces as well as, similar expected rigidity results (in particular, when $\lambda_X^{1,p} = (p-1)\frac{\pi_p^p}{\text{diam}^p}$ and $K=0$). We emphasize that sharp spectral gap estimates for p -Laplacian and the rigidity case when $K > 0$ already follow from the work of Cavalletti-Mondino [12, 13] (and in fact, in a more general setting of $CD^*(K, N)$ spaces) but our proofs have a different flavour (benefiting from the calculus tools available in $RCD^*(K, N)$ spaces) and also the rigidity in the case of non-negative Ricci is new.*

This article is organized as follows: In Section 2, we will briefly review some required background material; In order to demonstrate the strategy of the proof, in Section 3, we characterize the equality case of Zhong-Yang's estimates for eigenvalues of *drift Laplacian*; Section 4 is devoted to obtaining a crucial second order partial differential inequality (to which we will apply the strong maximum principle); In Section 5, we prove the $\frac{d^2}{2}$ -concavity of the potential function, f on suitable maximal domains; Section 6, discusses the short time existence, uniqueness and measure and distance preserving properties of the gradient flow of f on these maximal domains. In Section 7, we will demonstrate the claimed splitting phenomenon. Finally, in Section 8, we will complete the proof of Theorem 1.1.

We have decided to keep these notes, somewhat self contained and a reader who is already familiar with the basics of calculus in RCD spaces and equality case of Zhong-Yang estimates in manifolds might rather skip Sections 2 and 3.

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NOTATIONS

General optimal transport: Throughout this article, the metric measure space (X, d, m) will be considered to be *compact* unless otherwise stated. The space of probability measures on X is denoted by $\mathcal{P}(X)$ and the Wasserstein distance on this space is denoted by $W_2(\mu, \nu)$. π (bold π) will be denoting an optimal dynamical plan (which is a probability measure on $\text{Geo}(X)$). The set of such optimal plans for transporting measure μ to ν is denoted by $\text{OptGeo}(\mu, \nu)$. By the subscript x in π_x , we mean disintegration of π with respect to the evaluation at time 0 (i.e. with respect to the map, e_0).

Curvature-dimension conditions: For us, (X, d, m) will be a compact metric measure space satisfying *reduced Riemannian Ricci lower bounds* (a.k.a. $RCD^*(K, N)$ curvature-dimension conditions) unless otherwise specified. The Lott-Sturm-Villani curvature-dimension conditions will be denoted by $CD(K, N)$, Bacher-Sturm reduced curvature-dimension conditions by $CD^*(K, N)$ and the Bakry-Émery curvature-dimension conditions by $BE(K, N)$.

Laplacians and Hessians: $\text{Ch}(f)$ will denote the *Cheeger energy* of a function. The Laplacian associated to the Cheeger energy (via integration by parts), will be denoted by Δ_{Ch} . The measure valued Laplacian will be denoted by Δ . The usual Hessian of Riemannian geometry will be denoted by Hess while, the *Hessian* defined in an $RCD^*(K, N)$ space, X , will be Hess^X .

Γ – calculus: The *carré du champ* (a.k.a. "square field") operator associated to a diffusion operator, \mathbf{L} , will, in general, be denoted by Γ . Though, when the diffusion operator is Δ_{Ch} (which is the interesting case for us), we will use the more intuitive notation, $\langle \nabla f, \nabla g \rangle$ instead of $\Gamma(f, g)$. Γ_2 is defined by the usual iteration of Γ as in (8). Γ_2^X will be used to denote the Savaré's measure valued version of Γ_2 which is defined in (7). γ_2^X will denote the density of the absolutely continuous part of Γ_2^X with respect to the reference measure, m .

Function spaces: The space of Lipschitz functions on X will be denoted by $\mathbf{LIP}(X)$ which are also our usual test function space ($\text{Test}(X)$) and when restricting to an open domain, Ω , the test functions, $\text{Test}(\Omega)$ will be those Lipschitz functions with their supports inside Ω . The domain of the Cheeger energy, $\text{Ch}(\cdot)$, will be denoted by $\mathbb{D}(\text{Ch})$. We will see that these are the $W^{1,2}(X)$ Sobolev functions. $\mathbb{D}(\Delta_{\text{Ch}})$ will denote the domain of the Laplacian, Δ_{Ch} which is defined via integration by parts and using the carré du champ operator, Γ . The function space, $\mathbb{D}_+^\infty(\Delta_{\text{Ch}})$, is the following set of non-negative test functions:

$$\mathbb{D}_\infty^X(\Delta_{\text{Ch}}) := \{\varphi \geq 0 \in \mathbb{D}(\Delta_{\text{Ch}}) : \varphi, \Delta_{\text{Ch}}\varphi \in L^\infty(m)\}.$$

In Section 7, we will be stating the theorems regarding the function spaces, S_{loc}^2 . One can actually replace these with just Sobolev functions. For a precise definition of S_{loc}^2 spaces, see Gigli [24]. Another function space needed is

$$\mathbb{D}_\infty^X := \{g \in \mathbb{D}(\Delta_{\text{Ch}}) \cap L^\infty(m) : |\nabla g| \in L^\infty(m) \ \& \ \Delta_{\text{Ch}}g \in W^{1,2}(X)\}.$$

When we work with local version of these operators and function spaces, say in an open domain, Ω , the mentioned domains will be denoted in the following form:

$$\mathbb{D}_{(\cdot)}^{(\cdot)}((\text{the operator}), \Omega).$$

Finally, the domain of the measure values Laplacian, Δ is denoted by $\mathbb{D}(\Delta)$.

Specific Functions: Throughout this article, v will denote the unique eigenfunction relative to the first non-zero (Neumann) eigenvalue of the 1– dimensional model space. u will always denote an eigenfunction relative to the first non-zero eigenvalue of X ($\Delta_{\text{Ch}}u = -\lambda_1 u$). f will always denote the function $\sin^{-1}(u)$ which will proven to be harmonic in extremal spaces.

2. PRELIMINARIES

In a Riemannian manifold (M^n, g) , a lower bound on the Ricci curvature can be characterized solely in terms of the metric measure properties of the induced metric measure space, $(M, d_g, dvol_g)$, where d_g is the distance induced in M^n by the Riemannian metric g . It is, by now, well-known that, $\text{Ric}_{M^n} \geq K$ is equivalent to metric measure space, $(M, d_g, dvol_g)$, satisfying $CD(K, N)$ curvature-dimension conditions for $n \leq N$ and in the sense of Lott-Sturm-Villani (see the seminal

papers of Sturm [58, 59] and Lott-Villani [42]). These curvature-dimension conditions are stable under measured Gromov-Hausdorff convergence and hence, the limits of all such manifolds (called $(0, N)$ –Ricci limit spaces) also satisfy the Lott-Sturm-Villani’s curvature dimension conditions.

The class of $CD(K, N)$ spaces is however much bigger than the class of Ricci limit spaces (of Riemannian manifolds with dimension less than N and with $Ric \geq K$). In fact, there are Finsler manifolds that satisfy $CD(K, N)$ curvature-dimension conditions (see Ohta[47]) but from the seminal work of Cheeger-Colding in Ricci limit spaces (see [16, 17, 18, 19]), we know that Ricci limit spaces must be infinitesimally Hilbertian and hence, Finsler manifolds can not arise as Ricci limit spaces.

To exclude Finslerian spaces, Ambrosio-Gigli-Savaré [3] have introduced the notion of dimension-free Riemannian lower Ricci bounds (a.k.a. $RCD(K, \infty)$ curvature-dimension conditions) for compact metric measure spaces. Afterwards, Ambrosio-Gigli-Mondino-Rajala extended this notion to the non-compact metric spaces with σ -finite measures [1]. The *dimensional* Riemannian lower Ricci bounds for metric measure spaces (a.k.a. $RCD(K, N)$ curvature-dimension conditions) were later considered and investigated in Erbar-Kuwada-Sturm [22].

A *metric measure space* is a triple (X, d, m) consisting of a complete separable metric space, (X, d) , and a locally finite complete positive Borel measure, m , that is, $m(B) < \infty$ for any bounded Borel set B and $\text{supp } m \neq \emptyset$. We will always assume that $\text{supp}(m) = X$.

Roughly speaking, a $CD(K, N)$ metric measure space, (X, d, m) , is said to satisfy the Riemannian curvature-dimension conditions (for short, we will call it an $RCD(K, N)$ space) whenever, the associated weak Sobolev space $W^{1,2}$ is a Hilbert space. When $W^{1,2}$ is a Hilbert space, the underlying space, X , is said to be *infinitesimally Hilbertian*. In essence, infinitesimal Hilbertianity means that the heat flow and the Laplacian in these spaces (defined in [3]) are Linear. It is readily verified that Ricci limit spaces are in fact infinitesimally Hilbertian. It is also a well-known fact that an infinitesimally Hilbertian Finsler manifold has to be a Riemannian manifold which is a result of the Cheeger energy being a quadratic form. It is yet not known whether every $RCD(K, N)$ space is a Ricci limit space (not even known whether every Alexandrov space is so).

Bacher-Sturm [7] introduced reduced curvature-dimension conditions, $CD^*(K, N)$, in order to get better *local-to-global* and *tensorization* properties. Every $CD(K, N)$ space is also $CD^*(K, N)$; conversely, every $CD^*(K, N)$ space is proven to be a $CD(\frac{N-1}{N}K, N)$ space. In particular, $CD(0, N) = CD^*(0, N)$. As before, an infinitesimally Hilbertian $CD^*(K, N)$ space is said to be an $RCD^*(K, N)$ space. Recently, a structure theory for $RCD^*(K, N)$ spaces has been developed by Mondino-Naber [44]. They prove that the tangent space is unique almost everywhere. Also from Gigli-Mondino-Rajala [28], we know that almost everywhere, these unique tangent spaces are actually Euclidean namely isomorphic to $(\mathbb{R}^k, d_{Euc}, \mathcal{L})$ (k might vary point-wise). Using these developments, Kitabeppu and the author (see Kitabeppu-Lakzian [35]) have characterized $RCD^*(K, N)$ spaces when $N < 2$. It was shown that these low dimensional spaces are isometric to a circle or to a line segment.

As usual, a curve $\gamma : [0, l] \rightarrow X$ is called a *geodesic* if $d(\gamma(0), \gamma(l)) = \text{Length}(\gamma)$. We call (X, d) a *geodesic space* if for any two points, there exists a geodesic connecting them. A metric space (X, d) is said to be *proper* if every bounded closed set in X is compact. It is well-known that complete locally compact geodesic metric spaces are proper. In particular, compact spaces (that we are working with) are proper.

2.1. Lott-Sturm-Villani curvature-dimension conditions. Let (X, d, m) be a metric measure space and $\mathcal{P}(X)$, the set of all Borel probability measures. We denote by $\mathcal{P}_2(X)$, the set of all Borel probability measures with finite second moments.

For any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, the L^2 -Wasserstein distance is defined as

$$W_2(\mu_0, \mu_1) := \inf \left\{ \int_{X \times X} d(x, y)^2 dq(x, y) ; q \text{ is a coupling between } \mu_0, \mu_1 \right\}^{\frac{1}{2}}. \quad (4)$$

A measure, $q \in \mathcal{P}(X \times X)$, that realizes the infimum in (4), is called an *optimal coupling between μ_0 and μ_1* .

For every complete separable geodesic space, (X, d) , the L^2 -Wasserstein space, $(\mathcal{P}_2(X), W_2)$, is also a complete separable geodesic space. We denote by $\text{Geo}(X)$, the space of all constant speed geodesics from $[0, 1]$ to (X, d) with the sup norm and by $e_t : \text{Geo}(X) \rightarrow X$, the evaluation map for each $t \in [0, 1]$. It is known that any geodesic $(\mu_t)_{t \in [0, 1]} \subset \text{Geo}(\mathcal{P}_2(X))$ can be lifted to a measure $\pi \in \mathcal{P}(\text{Geo}(X))$, so that $(e_t)_* \pi = \mu_t$ for all $t \in [0, 1]$. Given two probability measures $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, we denote by $\text{OptGeo}(\mu_0, \mu_1)$ the space of all probability measures $\pi \in \mathcal{P}(\text{Geo}(X))$ such that $(e_0, e_1)_* \pi$ is an optimal coupling between μ_0 and μ_1 .

For given $K \in \mathbb{R}$ and $N \in [1, \infty)$, the distortion coefficients, $\sigma_{K, N}^{(t)}(\theta)$, are defined by

$$\sigma_{K, N}^{(t)}(\theta) := \begin{cases} \infty & \text{if } K\theta^2 \geq N\pi^2, \\ \frac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } 0 < K\theta^2 < N\pi^2, \\ t & \text{if } K\theta^2 = 0, \\ \frac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K\theta^2 < 0. \end{cases}$$

Definition 2.1 ((K, N) -convexity of functions). Suppose (X, d) is a geodesic space. A function $f : X \rightarrow \mathbb{R}$ is called (K, N) -convex if for any two points $x_0, x_1 \in X$ and a geodesic x_t , $0 \leq t \leq 1$ joining these points, one has

$$\begin{aligned} & \exp\left(-\frac{1}{N}f(x_t)\right) \\ & \geq \sigma_{K, N}^{(1-t)}(d(x_0, x_1)) \exp\left(-\frac{1}{N}f(x_0)\right) + \sigma_{K, N}^{(t)}(d(x_0, x_1)) \exp\left(-\frac{1}{N}f(x_1)\right). \end{aligned}$$

Definition 2.2 ($CD^*(K, N)$ curvature-dimension conditions). Let $K \in \mathbb{R}$ and $N \in [1, \infty)$. A metric measure space, (X, d, m) , is said to be a $CD^*(K, N)$ space if for any two measures $\mu_0, \mu_1 \in \mathcal{P}(X)$ with bounded support contained in $\text{supp } m$ and with $\mu_0, \mu_1 \ll m$, there exists a measure $\pi \in \text{OptGeo}(\mu_0, \mu_1)$ such that for every $t \in [0, 1]$ and $N' \geq N$ one has,

$$-\int \rho_t^{1-\frac{1}{N'}} dm \leq -\int \sigma_{K, N'}^{(1-t)}(d(\gamma_0, \gamma_1)) \rho_0^{-\frac{1}{N'}} + \sigma_{K, N'}^{(t)}(d(\gamma_0, \gamma_1)) \rho_1^{-\frac{1}{N'}} d\pi(\gamma),$$

where ρ_t for $t \in [0, 1]$, is the Radon-Nikodym derivative, $\frac{d(e_t)_* \mathbf{p}_1}{dm}$.

An infinitesimally Hilbertian metric measure space, (X, d, m) , that also satisfies $CD^*(K, N)$ curvature-dimension conditions is called an $RCD^*(K, N)$ space. Erbar-Kuwada-Sturm [22] gave another characterization of $RCD^*(K, N)$ spaces as follows.

Definition 2.3. Let (X, d, m) be a metric measure space. We say that (X, d, m) satisfies the *entropic curvature-dimension condition*, $CD^e(K, N)$, for $K \in \mathbb{R}$, $N \in (1, \infty)$ if for each pair

$\mu_0, \mu_1 \in \mathcal{P}_2(X, d, m)$ with finite entropy, there exists a constant speed geodesic, $(\mu_t)_{t \in [0,1]}$, connecting μ_0 to μ_1 such that for all $t \in [0, 1]$, the following holds

$$\exp\left(-\frac{1}{N} \text{Ent}(\mu_t)\right) \geq \sigma_{K,N}^{(1-t)}(W_2(\mu_0, \mu_1)) \exp\left(-\frac{1}{N} \text{Ent}(\mu_0)\right) + \sigma_{K,N}^{(t)}(W_2(\mu_0, \mu_1)) \exp\left(-\frac{1}{N} \text{Ent}(\mu_1)\right).$$

Theorem 2.4 (Theorem 3.17 in [22]). *Let (X, d, m) be an infinitesimally Hilbertian metric measure space. Then (X, d, m) is a $CD^*(K, N)$ space for $K \in \mathbb{R}$, $N \geq (1, \infty)$ if and only if (X, d, m) is a $CD^e(K, N)$ space.*

2.1.1. *Essentially non-branching.* Even though it is still unknown whether $RCD(K, N)$ spaces can, at all, contain branching geodesics, Rajala-Sturm [51] showed that an $RCD(K, N)$ space is essentially non-branching.

Definition 2.5 (essentially non-branching). A metric measure space, (X, d, m) , is said to be *essentially non-branching* if for any two absolutely continuous measures, $\mu, \nu \in \mathcal{P}(X)$, any optimal dynamical plan, $\pi \in \mathbf{OptGeo}(\mu, \nu)$, is concentrated on non-branching geodesics.

Rajala-Sturm's theorem (see Rajala-Sturm [51]) asserts that any strongly $CD(K, N)$ is essentially non-branching. The following is a corollary of their theorem.

Proposition 2.6 (essentially non-branching property, Rajala-Sturm [51]). *Any $RCD(K, \infty)$ is essentially non-branching.*

2.2. **Second order calculus in $RCD(K, N)$ spaces.** In this section, we briefly recall some of the second order calculus theory for $RCD(K, N)$ spaces that is developed mainly in Gigli [24, 23], Savaré [52] and Sturm [56].

2.2.1. *$RCD^*(K, N)$ spaces.* We denote the set of all Lipschitz functions in X by $\mathbf{LIP}(X)$. For every $f \in \mathbf{LIP}(X)$, the local Lipschitz constant at x , $|Df|(x)$, is defined by

$$|Df|(x) := \limsup_{y \rightarrow x} \frac{|f(x) - f(y)|}{d(x, y)},$$

when x is not isolated, otherwise $|Df|(x) := \infty$.

The *Cheeger energy* of a function $f \in L^2(X, m)$ is defined as

$$\text{Ch}(f) := \frac{1}{2} \inf \left\{ \liminf_{n \rightarrow \infty} \int_X |Df_n|^2 dm ; f_n \in \mathbf{LIP}(X), f_n \rightarrow f \text{ in } L^2 \right\}.$$

Set $\mathbb{D}(\text{Ch}) := \{f \in L^2(X, m) ; \text{Ch}(f) < \infty\}$. It is known that for any $f \in \mathbb{D}(\text{Ch})$, there exists $g =: |Df|_w \in L^2(X, m)$ such that $2\text{Ch}(f) = \int_X |Df|_w^2 dm$. We say that (X, d, m) is *infinitesimally Hilbertian* if the Cheeger energy is a quadratic form. Infinitesimal Hilbertianity is equivalent to the Sobolev space $W^{1,2}(X, d, m) := \{f \in L^2 \cap \mathbb{D}(\text{Ch})\}$ equipped with the norm $\|f\|_{1,2}^2 := \|f\|_2^2 + 2\text{Ch}(f)$ being a Hilbert space.

2.2.2. *Gradient flows in RCD spaces.* Suppose f is a Lipschitz potential function on a metric space (X, d) , then a *gradient flow trajectory* curve, $\gamma : [a, b] \rightarrow X$, of f is characterized by the following *energy dissipation* identity.

$$f(\gamma(t)) - f(\gamma(s)) = \frac{1}{2} \int_t^s |\dot{\gamma}| \, dr + \frac{1}{2} \int_t^s |Df|^2(\gamma(r)) \, dr \quad \forall t < s.$$

If the underlying space satisfies $RCD(K, N)$ curvature-dimension conditions, then for any $\frac{d^2}{2}$ -concave potential, f , the gradient flow exists, is unique and also is Lipschitz regular. For a very detailed discussion regarding gradient flows in metric spaces and in the Wasserstein space, see the excellent book, Ambrosio-Gigli-Savaré [2]. The most general result in this direction that we are aware of is the existence, uniqueness and Lipschitz regularity of the gradient flow for semi-convex (concave) potential functions by Sturm [55].

2.2.3. *The Laplacians, Δ_{Ch} and Δ .* Here, we define the measure valued Laplacian restricted to an open domain, $\Omega \subset X$ which we will denote by Δ^Ω (the definition is due to Gigli [24]). Then, letting $\Omega = X$, one retrieves Δ . A function $g \in W^{1,2}(X)$ is in the domain $\mathbb{D}(\Delta, \Omega)$ whenever there exists a signed Radon measure $\mu := \Delta^\Omega g$ on Ω such that

$$-\int_{\Omega} \langle \nabla h, \nabla g \rangle \, dm = \int_{\Omega} h \, d\mu,$$

holds for all Lipschitz functions, h with $\text{supp}(h) \subset \Omega' \subset \Omega$ for some open domain Ω' .

It is not hard to see that on letting, $\Omega = X$, one gets Δ^X (measure valued) is consistent with the (function valued) Laplacian, Δ_{Ch} obtained from the carré du champ, Γ via integration by parts (see Gigli [24]).

It is also easy to see that Δ is well-behaved under localization: if $g \in \mathbb{D}(\Delta, \Omega)$ and $\Omega' \subset \Omega$ open, then, $g|_{\Omega'} \in \mathbb{D}(\Delta, \Omega')$ and $\Delta^{\Omega'} g|_{\Omega'} = (\Delta^\Omega g)|_{\Omega'}$.

2.2.4. *The Gaussian estimates and Bakry-Émery contractions for the heat flow in $RCD^*(K, N)$ spaces.* One of the most important features (and in fact, a defining characteristic) of spaces satisfying $RCD^*(K, N)$ curvature-dimension bounds is that they enjoy a linear heat flow. In this section, we will briefly touch upon some basic facts regarding the heat flow and its properties in such spaces.

In an $RCD(K, N)$ space, X , the Heat flow, $\mathbf{H}_t : L^2(X) \rightarrow L^2(X)$ is a map such that for any $g \in L^2(X)$, $\varphi_t := \mathbf{H}_t(g) \in \mathbb{D}(\Delta)$, $t \geq 0$ is a continuous and locally absolutely continuous (for $t > 0$) solution of

$$\frac{d}{dt} \varphi_t = \Delta(\varphi_t) \quad \mathcal{L}^1 - a.e. \, t > 0 \quad \& \quad \mathbf{H}_0(g) = g.$$

Since the measure, m , is doubling under curvature-dimension bounds and since $RCD(K, N)$ spaces support $1 - 2$ weak local Poincaré inequalities (see Rajala [50, 49]), one can apply Sturm's existence and Gaussian estimates for heat kernels on metric measure spaces as proven in the seminal papers, Sturm [57, 53, 54]. In particular, with a little work, one can deduce the following estimate (for example, see Gigli [25, Section 4.1.2]).

$$\int_X d^n(x, y) \, \rho(t, x, y) \, dm(y) \leq C(n, N) \, t^{\frac{n}{2}} \quad \forall x \in X, \, t > 0. \quad (5)$$

It turns out that in an RCD space, the heat flow is in fact the gradient flow of the Cheeger energy (on the level of functions in L^2) or equivalently the gradient flow of the relative entropy in the Wasserstein space (on the level of measures). See Ambrosio-Gigli-Savaré [2] for more details.

Another important feature of the heat flow under curvature bounds and in particular, in $RCD(K, N)$ spaces, is the following Bakry-Ledoux gradient estimates (see Bakry-Ledoux [8] and Wang [61]).

Proposition 2.7 (Bakry-Ledoux [8], Wang [61]). *In an $RCD(K, N)$ space, one has*

$$|\nabla \mathbf{H}_t(g)|^2 \leq e^{-2Kt} \mathbf{H}_t(|\nabla g|^2) - \frac{1 - e^{-2Kt}}{NK} (\Delta_{\text{Ch}} \mathbf{H}_t(g))^2.$$

In the special case, $K = 0$, one retrieves the Bakry-Émery contraction property of the heat flow in $RCD(0, N)$ spaces which can also be easily deduced from the Bochner inequality by differentiation with respect to t . For $RCD(0, N)$ spaces, this contraction property reads as

$$|\nabla \mathbf{H}_t(g)| \leq \mathbf{H}_t(|\nabla g|^2) \quad m - a.e. x \in X \quad \& \quad \forall t \geq 0. \quad (6)$$

2.2.5. *Savaré's measure valued Γ_2^X .* For any $g \in \mathbb{D}_\infty^X$, Savaré's measure valued $\Gamma_2^X(g)$ is defined by

$$\Gamma_2^X(g) := \frac{1}{2} \Delta |\nabla g|^2 - \langle \nabla g, \nabla \Delta_{\text{Ch}} g \rangle m. \quad (7)$$

One can write $\Gamma_2^X(g) = \gamma_2^X m + \gamma_{\text{sing}}$ where, $0 \leq \gamma_{\text{sing}} \perp m$.

Savaré [52] proved the following very useful chain rule formula :

Lemma 2.8 (chain rule for γ_2^X , Savaré [52]). *Let g^1, g^2, \dots, g^n be in \mathbb{D}_∞^X and let $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^3 smooth map. Put $\mathbf{g} := (g^i)_{i=1}^n$. Then (using Einstein summation conventions),*

$$\begin{aligned} \gamma_2^X(\Phi(\mathbf{g})) &= \Phi_{x_i}(\mathbf{g}) \Phi_{x_j}(\mathbf{g}) \gamma_2^X(g^i, g^j) + 2\Phi_{x_i}(\mathbf{g}) \Phi_{x_i x_k}(\mathbf{g}) \text{Hess}^X(g^i)(g^j, g^k) \\ &\quad + \Phi_{x_i x_k}(\mathbf{g}) \Phi_{x_j x_h}(\mathbf{g}) \langle \nabla g^i, \nabla g^j \rangle \langle \nabla g^k, \nabla g^h \rangle. \end{aligned}$$

2.2.6. *Hessian and Sturm's Hessian estimate.* Suppose one has a well-defined carré du champ operator, Γ , defined on a suitable algebra, \mathcal{A} , of functions (which roughly speaking, plays the role of a tangent space). Then, the Hessian of a function $f \in \mathcal{A}$, *a la Sturm* [56], (and also Gigli [23]) is defined to be

$$\text{Hess}^X(f)(\varphi, \psi)(x) := \frac{1}{2} [\Gamma(\varphi, \Gamma(f, \psi)) + \Gamma(\psi, \Gamma(f, \varphi)) - \Gamma(f, \Gamma(\varphi, \psi))].$$

In an $RCD^*(K, N)$ space, X , using our notations, the Hessian takes the following form which is reminiscent of the closed formulae for Hessian on Riemannian manifolds:

$$\text{Hess}^X(f)(\varphi, \psi)(x) := \frac{1}{2} [\langle \nabla \varphi, \nabla \langle \nabla f, \nabla \psi \rangle \rangle + \langle \nabla \psi, \nabla \langle \nabla f, \nabla \varphi \rangle \rangle - \langle \nabla f, \nabla \langle \nabla \varphi, \nabla \psi \rangle \rangle].$$

The following are the standard chain rules satisfied by Δ_{Ch} and Hess^X .

Proposition 2.9.

$$f^{-p} \text{Hess}^X(f^p)(\varphi, \varphi) = p \text{Hess}^X(\ln f)(\varphi, \varphi) + p^2 \langle \nabla \ln f, \nabla \varphi \rangle,$$

and

$$\Delta_{\text{Ch}}(\varphi \circ g) = \varphi' \circ g \Delta_{\text{Ch}} g + \varphi'' \circ g |\nabla g|^2.$$

Proof.

For example, see Gigli [24] for a proof. ■

There is a very nice and powerful estimate on the $\text{Hess}^X(u)(g, h)$ proven by Sturm [56]. The following is a special case.

Proposition 2.10 (Hessian estimate of Sturm [56]). *Let (X, d, m) be an $RCD^*(K, N)$ space with $K \in \mathbb{R}$ and $N \geq 1$, then for $u, f, g \in \mathbb{D}_\infty^X$, one has*

$$\begin{aligned} & 2 \left[\text{Hess}^X(u)(g, h) - \frac{1}{N} \langle \nabla g, \nabla h \rangle \Delta u \right]^2 \\ & \leq \left[\Gamma_2(u) - \frac{1}{N} (\Delta u)^2 - K |\nabla u|^2 \right] \cdot \underbrace{\left[\frac{N-2}{N} \langle \nabla g, \nabla h \rangle^2 + |\nabla g|^2 |\nabla h|^2 \right]}_{\Lambda}. \end{aligned}$$

Notice that, $\Lambda > 0$ in non-trivial cases.

2.2.7. Regularity of Sobolev functions. From Rajala [49] and Ambrosio-Gigli-Savaré [3], we have the following Lipschitz regularity for Sobolev functions.

Proposition 2.11. *Let (X, d, m) be an $RCD^*(K, N)$ space, then any Sobolev function, $g \in W^{1,2}(X)$, admits a Lipschitz representative.*

So, in the virtue of Proposition 2.11, we will always assume the Sobolev functions (which turn out to be the elements in the domain of Δ_{Ch}) to be Lipschitz continuous.

2.2.8. Strong maximum principle in metric spaces. Björn-Björn [10] proved the following strong maximum principle

Proposition 2.12 (strong maximum principle). *Suppose (X, d, m) is a doubling metric measure space that supports a $1 - 2$ weak local Poincaré inequality. Let $g \in S_{\text{loc}}^2(X)$ be continuous and such that*

$$\int_{\Omega} |\nabla g|^2 \, dm \leq \int_{\Omega} |\nabla(g + h)|^2 \, dm,$$

holds for any $h \in \text{Test}(X)$ and bounded open set, Ω with $\text{supp}(h) \subset \Omega \subset X$. If g attains its maximum inside Ω , then g is constant in Ω .

In the setting of $RCD(K, N)$ spaces, the strong maximum principle will imply the following.

Proposition 2.13. *Let Ω be a bounded open set and let g be a sub-minimizer of the Cheeger energy, $\text{Ch}(\cdot)$. If g attains its maximum inside Ω , then g is constant on Ω .*

Alternatively, for our purposes, one can also use the elliptic maximum principle for general Dirichlet form spaces as is presented in Grigor'yan-Hu [30, Proposition 4.6].

2.3. Bakry-Émery curvature-dimension conditions. Here, we briefly recall the definition of Bakry-Émery $BE(K, N)$ curvature-dimension conditions in the case that the underlying space is already an $RCD^*(K, N)$ space. Actually, it turns out that, these notions coincide provided that the underlying space is infinitesimally Hilbertian.

In an $RCD^*(K, N)$ space, the Cheeger energy, Ch^X is a *strongly local* and *regular* Dirichlet form. The set, $\text{Lip}(X)$, of Lipschitz functions is dense in the Sobolev space, $W^{1,2}(X)$ (see Gigli [24] for definitions). This implies that there is a carré du champ operator, Γ , associated to this Dirichlet form. We suppress the Γ notation and simply point out that we have the following symmetric bi-linear form

$$\langle \nabla(\cdot), \nabla(\cdot) \rangle : W^{1,2}(X) \times W^{1,2}(X) \rightarrow L^1(m).$$

Define the operator, Γ_2 , by

$$\Gamma_2[u, v; \varphi] := \int_X \langle \nabla u, \nabla v \rangle \Delta_{\text{Ch}} \varphi dm - \int_X [\langle \nabla u, \nabla \Delta_{\text{Ch}} v \rangle + \langle \nabla \Delta_{\text{Ch}} u, \nabla v \rangle] \varphi dm,$$

for $u, v \in \mathbb{D}(\Gamma_2)$ and $\varphi(\geq 0) \in L^2(m) \cap L^\infty(m)$ where,

$$\mathbb{D}(\Gamma_2) := \{ \varphi \in \mathbb{D}(\Delta_{\text{Ch}}) : \Delta_{\text{Ch}} \varphi \in W^{1,2}(X) \}.$$

Definition 2.14. We say that the cheeger energy, Ch satisfies (K, N) Bakry-Émery curvature conditions (in short, $BE(K, N)$ condition) for $K \in \mathbb{R}$ and $N \in [1, \infty]$ whenever for every $u \in \mathbb{D}(\Gamma_2)$ and $\varphi(\geq 0) \in L^2(m) \cap L^\infty(m)$, there holds

$$\Gamma_2(u; \varphi) \geq K \int_X |\nabla u|^2 \varphi dm + \frac{1}{N} \int_X (\Delta_{\text{Ch}} u)^2 dm.$$

2.3.1. Equivalence of Bakry-Émery and RCD^* curvature-dimension bounds. One crucial fact that will enables us to benefit from the tools available for both $BE(K, N)$ and $RCD^*(K, N)$ spaces is the following important equivalence theorem proven by Erbar-Kuwada-Sturm [22] and Ambrosio-Gigli-Savaré [4].

Proposition 2.15 (Erbar-Kuwada-Sturm [22] and Ambrosio-Gigli-Savaré [4]). *Let $K \in \mathbb{R}$ and $N \geq 1$. We have the following*

$$\text{infinitesimally Hilbertian} + BE(K, N) \iff RCD^*(K, N).$$

2.4. Non-smooth splitting theorem. In this section, we outline Gigli's strategy in Gigli [25, 26] for proving the splitting theorem for $RCD(0, N)$ spaces.

2.4.1. The harmonic potential, \mathbf{b} . Suppose X is complete and contains a line, $\gamma : (-\infty, +\infty) \rightarrow X$. Then, just as in the smooth case, one can define the Busemann functions, \mathbf{b}_\pm as follows:

$$\mathbf{b}_\pm(x) := \lim_{t \rightarrow \pm\infty} d(\gamma(t), x) - t.$$

It is easy to see that, \mathbf{b}_\pm is 1-Lipschitz thus, $|\nabla \mathbf{b}_\pm| \leq 1$. Then, applying the Laplacian comparison Theorem (for example see [24, Corollary 5.15]), one can prove that \mathbf{b}_\pm are *sub-harmonic* functions.

Applying the *strong maximum principle* to the sub-harmonic function, $g := \mathbf{b}_+ + \mathbf{b}_-$ (it is easy to see that g is 1-Lipschitz and non-positive and vanishing on the line, γ) one deduces that indeed, $g = 0$ on X and hence, $\mathbf{b}_+ = -\mathbf{b}_-$ which in turn, implies that $\Delta g = 0$.

2.4.2. $\frac{d^2}{2}$ -concavity of \mathbf{b} and its gradient flow. It is also straightforward to prove that multiples of the Busemann function are $c := \frac{d^2}{2}$ -concave, which means that $(a\mathbf{b})^{cc} = a\mathbf{b}$ where, for any φ , the Legendre transform, φ^c , is defined by

$$\varphi^c(y) := \inf_{x \in X} \left[\frac{d(x, y)^2}{2} - \varphi(x) \right].$$

(One should think of the Legendre transform in linear algebra).

The $\frac{d^2}{2}$ -concavity of \mathbf{b} , will then guarantee the existence and uniqueness of the gradient flow of \mathbf{b} . One can also show that the gradient flow trajectories of \mathbf{b} are in fact, geodesics. In the smooth setting, this would mean that ∇f is parallel (which would follow from $\text{Hess}(\mathbf{b}) = 0$)

Using the harmonicity of \mathbf{b} , then, one can prove that the gradient flow of \mathbf{b} preserves the measure. The harmonicity, will also imply that \mathbf{b} is invariant under the *heat flow* in X which in turn, means that the *Dirichlet energy* is preserved under the gradient flow of \mathbf{b} . Preservance of *Dirichlet energy* in particular, implies that the distances are being preserved under the gradient flow of \mathbf{b} .

2.4.3. *Splitting of the underlying space.* Once one has proven the above facts about the gradient flow of the \mathbf{b} . It can be shown that taking $X' := \mathbf{b}^{-1}(0)$, one can show that (X, d, m) isometrically splits off a line namely,

$$(X, d, m) \stackrel{\text{isom}}{\cong} (\mathbb{R} \times X', d_{Euc} \oplus d', \mathcal{L}^2 \otimes m'),$$

where, d' and m' are defined canonically.

2.5. **Gradient comparison and sharp spectral gap for $RCD^*(K, N)$ spaces.** We recall a crucial gradient comparison for u which will be used extensively in the later sections. This gradient comparison has been proven by Ketterer [34] for $K > 0$ and by Jiang-Zhang [32] for all $K \in \mathbb{R}$.

Proposition 2.16 (Ketterer [34] and Jiang-Zhang [32]). *Let (X, d, m) be a compact $RCD^*(K, N)$ space. Let λ_1 be the first non-zero eigenvalue of X and suppose $\lambda_1 > \max \left\{ 0, \frac{lK}{l-1} \right\}$ for some $l \geq N$. Let u be an eigenfunction with respect to λ_1 and let v be a Neumann eigenfunction for the $\hat{\lambda}(N, K, d)$. If $[\min u, \max u] \subset [\min v, \max v]$, then*

$$|\nabla u|^2(x) \leq (v' \circ v^{-1})^2(u(x)) \quad \text{for } m - a.e. x \in X.$$

Pick p_{-1} and p_{+1} such that $u(p_{\pm}) = \pm 1$. Let $\gamma : [0, l] \rightarrow M^n$ be a unit speed minimal geodesic joining p_{-1} and p_{+1} . Such geodesics joining from S_{-1} to S_{+1} are called *horizontal geodesics*.

Using the above gradient estimate for eigenfunctions, one can prove the following key lemma.

Lemma 2.17. *Suppose (X, d, m) is a compact $RCD(0, N)$ space with $\text{diam } X = \pi$ and with $\lambda_1 = 1$. Then, $\max u = -\min u = \max v = -\min v = 1$. Furthermore, for any two points p_- and p_+ with $u(p_{\pm}) = \pm 1$ and a geodesic, γ joining them, we have $\text{Length}(\gamma) = \pi$ and $|\nabla u|^2 + u^2 = 1$ on γ .*

Proof.

From the proof of Bakry-Qian [9] and Jiang-Zhang [32], we know that, for the extremal spaces and since $\text{diam}(X) = \pi$, we have $\max u = -\min u = \max v = -\min v = 1$. Also, in the 1-

dimensional model space, $v(t) = \sin(t)$ hence, $[v'(t)]^2 = 1 - v^2$. Therefore, from the gradient comparison, we get

$$|\nabla u|^2 \leq (v' \circ v^{-1}(u)) = 1 - u^2,$$

and that the equality is achieved on γ . Let $\alpha = |\nabla u|^2 + u^2$. From above we know that $\alpha \leq 1$ and on γ , $\alpha = 1$. Then,

$$\begin{aligned} \pi = d &\geq \int_0^l |\dot{\gamma}| dt && \geq \int_{|u \circ \gamma| \neq 0} |\dot{\gamma}| dt \\ &&& \geq \int_{|u \circ \gamma| \neq 0} \frac{|u \circ \gamma|}{|\nabla u|(\gamma(t))} dt = \int_{-1}^1 \frac{1}{|\nabla u|} du \\ &&& \geq \int_{-1}^1 \frac{1}{(v' \circ v^{-1}(u))} du \geq \pi. \end{aligned}$$

This means that on γ , one has:

$$|\nabla u|(\gamma(t)) = (v' \circ v^{-1}(u)) = 1 - u^2(\gamma(t)), \quad \mathcal{L}^1 - a.e. \ t.$$

■

3. CHARACTERIZATION OF EQUALITY FOR EIGENVALUES OF THE DRIFT LAPLACIAN

In this section, to demonstrate the strategy of the proof, we will prove the rigidity of the equality case of Zhong-Yang's spectral gap estimates for the drift Laplacian. This proof entails easy adjustments in the classic proof of Hang-Wang [31]. Since, we have not found a proof in the literature, we present a brief one here.

Let (M^n, g) be a smooth Riemannian manifold. Associated to a smooth vector field, χ , on M^n , one defines the drift Laplacian

$$\Delta_\chi = \Delta - D_\chi.$$

Suppose $n < N$. The N -Bakry-Émery Ricci tensor associated to the vector field χ is defined as

$$\text{Ric}_\chi^N = \text{Ric} + \frac{1}{2} \mathcal{L}_\chi g - \frac{\chi^\# \otimes \chi^\#}{N - n}.$$

The Bakry-Émery curvature-dimension condition $(BE(K, N))$, then, becomes

$$\text{Ric}_\chi^N \geq Kg,$$

as tensors.

Remark 3.1. In the special case $\chi = \nabla W$ for some smooth potential, W ; the triple $(M^n, g, e^{-f} d\text{vol}_g)$ is also called a Bakry-Émery manifold or a manifold with density. In this case, the drift Laplacian takes the form

$$\Delta_W = \Delta + \nabla f \cdot \nabla.$$

Also the N -Bakry-Émery Ricci tensor becomes

$$\text{Ric}_f^N = \text{Ric} + \text{Hess}(W) - \frac{dW \otimes dW}{N - n}.$$

The Bochner formula, for the drift Laplacian, takes the form

$$\frac{1}{2} \Delta_\chi |\nabla f|^2 = |\text{Hess } f|^2 + \text{Ric}(\nabla f, \nabla f) + \frac{1}{2} \mathcal{L}_\chi g(\nabla f, \nabla f) + \langle \nabla \Delta_\chi f, \nabla f \rangle,$$

and with a little work, one can prove the following Bochner inequality

$$\frac{1}{2} \Delta_\chi |\nabla f|^2 \geq \frac{(\Delta u)^2}{N} + K |\nabla f|^2 + \langle \nabla \Delta_\chi f, \nabla f \rangle.$$

When $\chi = \nabla W$, from the work of Bakry-Qian [9] or Andrews-Clutterbuck [6], one has the usual eigenvalue comparison estimates (1). In particular, we know that $\text{Ric}_f^N \geq 0$ implies that $\lambda_1 \geq \frac{\pi^2}{d^2}$ where, d is the diameter of M^n .

Without assuming that χ admits a potential function, the eigenvalue comparison follows from the recent more general results for metric measure spaces with Riemannian lower Ricci curvature bounds as is proven in Jiang-Zhang [32].

Suppose $\lambda_1 = \frac{\pi^2}{d^2}$. Our goal is to prove that M^n has to be one dimensional which then implies that M^n is either a circle of radius $\frac{d}{2}$ or a segment of length d .

The idea is to prove that, there exist a finite set S and an $n - 1$ dimensional Riemannian manifold (N, h) such that

$$M^n \setminus S \stackrel{\text{isom}}{\cong} \left(\left(\frac{d}{2}, -\frac{d}{2} \right) \times N, dt^2 \oplus h \right),$$

and indeed, we want to show that M^n splits isometrically quite similar to Cheeger-Gromoll's splitting phenomenon.

First by rescaling, we can assume that $\text{diam}(M^n) = \pi$ and $\lambda_1 = 1$. Suppose u is an eigenfunction associated to $\lambda_1 = 1$. From the proof of Bakry-Qian [9], we know that

$$-\min v = \max v = \max u = -\min u = 1,$$

and, also we have the gradient comparison

$$|\nabla u|^2 \leq (v' \circ v^{-1})^2(u),$$

where, v is the eigenfunction of the appropriate 1-dimensional model space (see (1)-(3)).

As before, for $p_{\pm 1} \in S_{\pm 1}$ and a geodesic γ joining them, one has $\text{Length}(\gamma) = \pi$ and that on γ ,

$$|\nabla u|^2(\gamma(t)) = (v' \circ v^{-1}(u))^2 = 1 - u^2(\gamma(t)).$$

(See the key Lemma 2.17).

The following key lemma provides us with a nice PDI:

Lemma 3.1. *Let u be as in above. $\alpha := |\nabla u|^2 + u^2$ and $\beta := |\nabla u|^2 - u^2$. Set $M_u := \{x \in M^n : \nabla u(x) \neq 0\}$. Then, on M_u , the following PDI holds.*

$$\underbrace{\frac{1}{2} \Delta \alpha - \frac{1}{2} D_\chi \alpha}_{\Delta_\chi \alpha} - \frac{\nabla \alpha \cdot \nabla \beta}{4 |\nabla u|^2} \geq \text{Ric}_\chi^N. \quad (8)$$

Proof.

As 1-forms we have

$$\nabla_{(\cdot)} |\nabla u|^2 = 2 \langle \nabla_{(\cdot)} \nabla u, \nabla u \rangle.$$

Therefore,

$$\nabla \alpha \cdot \nabla \beta = \left| \nabla |\nabla u|^2 \right|^2 - |\nabla u^2|^2 \leq 4 |\text{Hess } u|^2 |\nabla u|^2 - 4 u^2 |\nabla u|^2,$$

so,

$$\frac{\nabla \alpha \cdot \nabla \beta}{4 |\nabla u|^2} \leq |\text{Hess } u|^2 - u^2.$$

Now, with the aid of the Bochner formula and chain rule for Δ_χ , we compute

$$\begin{aligned} \frac{1}{2} \Delta_\chi \alpha &= \frac{1}{2} \Delta_\chi (|\nabla u|^2) + \frac{1}{2} \Delta_\chi (u^2) \\ &= |\text{Hess } u|^2 + \text{Ric}(\nabla u, \nabla u) + \frac{1}{2} \mathcal{L}_\chi g(\nabla u, \nabla u) + \langle \nabla \Delta_\chi u, \nabla u \rangle + \frac{1}{2} \Delta_\chi (u^2) \\ &= |\text{Hess } u|^2 - u^2 + \text{Ric}(\nabla u, \nabla u) + \frac{1}{2} \mathcal{L}_\chi g(\nabla u, \nabla u) \\ &\geq \frac{\nabla \alpha \cdot \nabla \beta}{4 |\nabla u|^2} + \text{Ric}_\chi^N(\nabla u, \nabla u). \end{aligned}$$

■

Let Y be the set on which α attains its maximum i.e.

$$Y := \left\{ x : |\nabla u|^2 + |u|^2 = 1 \right\} \neq \emptyset.$$

It is obvious that Y is closed. Applying the strong maximum principle to the PDI, (8), one observes that Y is also open. So, we must have $Y = X$. This means that $\alpha \equiv 1$ on M^n .

Lemma 3.2. *On M_u , define the vector field $\zeta := |\nabla u|^{-1} \nabla u$. Then, $\text{Hess } u$ is given by*

$$\text{Hess } u = -u \nabla_{(\cdot)} u \otimes \nabla_{(\cdot)} u.$$

Proof.

By Lemma 3.1 and constancy of α , we have

$$|\text{Hess } u|^2 = u^2.$$

From the standard Riemannian geometry, we know that

$$\text{Hess } u(\nabla g, \nabla h) = \frac{1}{2} [\langle \nabla g, \nabla \langle \nabla u, \nabla h \rangle \rangle + \langle \nabla h, \nabla \langle \nabla u, \nabla g \rangle \rangle - \langle \nabla u, \nabla \langle \nabla g, \nabla h \rangle \rangle],$$

hence,

$$\text{Hess } u(\nabla u, \nabla g) = \frac{1}{2} \langle \nabla g, \nabla |\nabla u|^2 \rangle = \frac{1}{2} \langle \nabla g, \nabla (1 - u^2) \rangle = -u \langle \nabla u, \nabla g \rangle,$$

and in particular,

$$\text{Hess } u(\zeta, \zeta) = -u.$$

Now, (9) and (9) imply that in an orthonormal frame $\{\zeta, e_2, \dots, e_n\}$, we have

$$\text{Hess } u(e_i, e_i) = 0 \quad i = 2, \dots, n,$$

and we have the desired result. ■

Now let $f := \sin^{-1}(u)$ and observe that $\zeta = \nabla f = |\nabla u|^{-1} \nabla u$. We have

$$\nabla f = (1 - u^2)^{-\frac{1}{2}} \nabla u,$$

and therefore,

$$\begin{aligned} \text{Hess } f &= u(1 - u^2)^{-\frac{3}{2}} \nabla u \otimes \nabla u + (1 - u^2)^{-\frac{1}{2}} \text{Hess } u \\ &= u(1 - u^2)^{-\frac{3}{2}} \nabla u \otimes \nabla u - u \nabla_{(\cdot)} \zeta \otimes \nabla_{(\cdot)} \zeta = 0. \end{aligned}$$

Notice that ∇u vanishes on a finite set, S (in fact by, Hang-Wang [31], one has $\#(S) \leq 4$). Nevertheless, ζ can be extended to the entire M^n . Now f being Hessian free means that $\zeta = |\nabla u|^{-1} \nabla u$ is a parallel vector field in M^n . This means that the tangent bundle splits off a 1-dimensional subspace. Now, in the virtue of the de Rham decomposition theorem, this implies that the underlying manifold also splits isometrically. This is in fact quite similar to the Cheeger-Gromoll's splitting phenomenon. Recall that Cheeger-Gromoll's splitting theorem is proven by showing that the Beusmann function, b is Hessian free and then using the de Rham decomposition theorem (see Cheeger [15]).

If $N(= u^{-1}(0))$ is connected, we have

$$\pi = \text{diam}(M^n) = (\pi^2 + \text{diam}(N)^2)^{\frac{1}{2}},$$

which means $\text{diam}(N) = 0$ and hence, $N = pt$ and so M^n is one dimensional and we are done. Otherwise, repeating the same argument for connected components of N tells us that N consists of isolated points and we have M^n is a disjoint union of circles or disjoint union of segments (which is allowed if allow for manifolds that are not connected) and we are done.

4. THE APPROPRIATE PDI IN $RCD(0, N)$ SPACES

Our goal in this section, is to trace our footsteps in Section 3 and make appropriate adjustments.

In a nutshell, we will prove that f is still harmonic and Hessian free in a suitable sense. Then in Sections 5 - 8, we will adapt the Gigli's approach in the proof of the Splitting theorem in non-smooth setting, to our setting and consequently show that the underlying space has to split isometrically (and in fact, be 1-dimensional.)

Following the proof of spectral gap estimates in $RCD(K, N)$ spaces as in Jiang-Zhang [32], we again notice that when $\lambda_1 = 1 = \lambda(0, N, \pi)$, we must have

$$-\min v = \max v = \max u = -\min u = 1,$$

where u and v are as before.

Arguing as in the last section, we know that $\alpha \leq 1$ and the equality is achieved on a minimal geodesic γ connecting p_{-1} to p_{+1} .

Recall Savaré's measure valued Γ_2^X from Section 2.2.5, which is defined as:

$$\Gamma_2^X(u) := \frac{1}{2} \Delta |\nabla u|^2 - \langle \nabla u, \nabla \Delta u \rangle m = \gamma_2^X(u) + \gamma_{sing}.$$

Lemma 4.1. *On $X_u := \{x \in X : |\nabla u| \neq 0\}$, α satisfies*

$$\frac{1}{2} \Delta_{\text{Ch}} \alpha - \frac{\langle \nabla \alpha, \nabla \beta \rangle}{4 |\nabla u|^2} \geq 0.$$

Proof.

We start by computing $\Gamma_2^X(\alpha)$ and $\Gamma_2^X(u)$:

$$\Gamma_2^X(\alpha) := \frac{1}{2} \Delta \alpha - \langle \nabla \alpha, \nabla \Delta \alpha \rangle m = \gamma_2^X(\alpha) + \gamma_{sing},$$

and

$$\Gamma_2^X(u) := \frac{1}{2} \Delta |\nabla u|^2 + \langle \nabla u, \nabla u \rangle m = \gamma_2^X(u).$$

As before, we compute

$$\begin{aligned}\langle \nabla \alpha, \nabla \beta \rangle &= \left\langle \nabla \left(|\nabla u|^2 + u^2 \right), \nabla \left(|\nabla u|^2 - u^2 \right) \right\rangle \\ &= \left\langle \nabla |\nabla u|^2, \nabla |\nabla u|^2 \right\rangle - \left\langle \nabla u^2, \nabla u^2 \right\rangle = \left\langle \nabla |\nabla u|^2, \nabla |\nabla u|^2 \right\rangle - 4u^2 |\nabla u|^2.\end{aligned}$$

Therefore, on $X_u := \{x : |\nabla u| \neq 0\}$, we get:

$$\langle \nabla \alpha, \nabla \beta \rangle = \frac{\left\langle \nabla |\nabla u|^2, \nabla |\nabla u|^2 \right\rangle}{4 |\nabla u|^2} - u^2 = \frac{|\nabla |\nabla u|^2|^2}{4 |\nabla u|^2} - u^2.$$

From [52, Theorem 3.4], we can deduce:

$$\frac{\langle \nabla \alpha, \nabla \beta \rangle}{4 |\nabla u|^2} = \frac{|\nabla |\nabla u|^2|^2}{4 |\nabla u|^2} - u^2 \leq \gamma_2^X(u) - u^2.$$

Hence, in the sense of distributions, we get:

$$\frac{1}{2} \Delta_{\text{Ch}} |\nabla u|^2 = \gamma_2^X(u) - \langle \nabla u, \nabla u \rangle \geq \frac{\langle \nabla \alpha, \nabla \beta \rangle}{4 |\nabla u|^2} - |\nabla u|^2 + u^2,$$

and consequently,

$$\begin{aligned}\frac{1}{2} \Delta_{\text{Ch}} \alpha &= \frac{1}{2} \Delta_{\text{Ch}} \left(|\nabla u|^2 \right) + \frac{1}{2} \Delta_{\text{Ch}} (u^2) \\ &\geq \frac{\langle \nabla \alpha, \nabla \beta \rangle}{4 |\nabla u|^2} - |\nabla u|^2 + u^2 + |\nabla u|^2 - u^2.\end{aligned}$$

Therefore,

$$\frac{1}{2} \Delta_{\text{Ch}} \alpha - \frac{\langle \nabla \alpha, \nabla \beta \rangle}{4 |\nabla u|^2} \geq 0,$$

is true in the sense of distributions. ■

Lemma 4.2. *The set $Y := \{x \in X : |\nabla u|^2 + u^2 = 1\}$ is open.*

Proof.

Applying the strong maximum principle for sub-minimizers of the Cheeger energy (see Proposition 2.12), we deduce that if α attains its maximum (which is $= 1$) at a point p , then $\Delta_{\text{Ch}} \alpha(p) \leq 0$. So, arguing as before, we get Y is open. ■

Y is obviously closed since α is continuous (and in fact, Lipschitz). This means that $Y = X$ and therefore, $\alpha \cong 1$.

Straight from the definition of Hessian in RCD setting (see (8)). One computes

$$\text{Hess}^X(u)(u, g) = \frac{1}{2} \left\langle \nabla |\nabla u|^2, \nabla g \right\rangle = -u \langle \nabla u, \nabla g \rangle,$$

and in particular,

$$\text{Hess}^X(u)(f, f) = -u,$$

where, $f = \sin^{-1}(u)$ as before.

Using the chain rule (see [24]), we also compute

$$\begin{aligned}\text{Hess}^X(f)(f, g) &= \frac{1}{2} \left\langle \nabla |\nabla f|^2, \nabla g \right\rangle = \frac{1}{2} \left\langle \nabla \left[(\sin^{-1})'(u)^2 |\nabla u|^2 \right], \nabla g \right\rangle \\ &= \langle \nabla 1, \nabla g \rangle = 0.\end{aligned}$$

Also integrating $\gamma_2^X - u^2 \geq 0$, we get

$$0 \leq \int_X \gamma_2^X - u^2 \, dm = \int_X \frac{1}{2} \Delta |\nabla u|^2 \, dm = 0,$$

which means that $\gamma_2^X(u) = u^2$ a.e. Using the chain rule for Laplacian (see Proposition 2.9), we have

$$\begin{aligned} \Delta_{\text{Ch}}(f) &= \Delta_{\text{Ch}}[\sin^{-1}(u)] = (\sin^{-1})'(u) \Delta_{\text{Ch}} u + (\sin^{-1})''(u) u |\nabla u|^2 \\ &= \frac{-u}{\sqrt{1-u^2}} + u(1-u^2)^{-\frac{3}{2}} |\nabla u|^2 \\ &= \frac{-u}{\sqrt{1-u^2}} = 0. \end{aligned}$$

Remark 4.1. Ketterer [34] uses the Hessian identity $\text{Hess}^X(u)(g, h) = -\frac{KN}{N-1} u \langle \nabla g, \nabla h \rangle$ for $K > 0$ and his maximum diameter theorem (see Ketterer [33]) to prove the Obata's warped splitting. There, the Hessian identity has been proven by using Sturm's Hessian estimates (see Proposition 2.10). A posteriori, it will become obvious that $\text{Hess}^X(u)(g, h) \equiv -u \langle \nabla g, \nabla h \rangle$ in our case, but using the Sturm's Hessian estimates (the same way as in Ketterer [34]), one gets

$$\begin{aligned} &\frac{2}{\Lambda} \left[\text{Hess}^X(u)(f, g) - \frac{1}{N} \langle \nabla f, \nabla g \rangle \Delta u \right]^2 \\ &\leq \left[\Gamma_2(u) - \frac{1}{N} (\Delta u)^2 - R_N(u) \right] \leq \Gamma_2(u), \end{aligned}$$

thus,

$$\begin{aligned} \mathbf{A} &:= \int_X \frac{2}{\Lambda} \left[\text{Hess}^X(u)(f, g) - \frac{1}{N} \langle \nabla f, \nabla g \rangle \Delta u \right]^2 \, dm \\ &\leq \int_X \gamma_2^X(u) \, dm \\ &= \int_X \Gamma_2^X(u) \, dm \\ &= \int_X \frac{1}{2} \Delta |\nabla u|^2 + (\Delta u)^2 \, dm \\ &= \frac{N-1}{N} \int_X u^2 \, dm \\ &= \frac{N-1}{N} \int_X |\nabla u|^2 \, dm \leq \frac{N-1}{N} \int_X 1 - u^2 \, dm, \end{aligned}$$

which does not lead to $A = 0$ if $N > 1$. Using Sturm's Hessian estimates, in a different manner, we have only been able to prove the following, which gives an inequality between the norms.

Lemma 4.3. $|\text{Hess}^X(u)(g, h)| \leq |u \langle \nabla g, \nabla h \rangle|$.

Proof.

If $N = 1$, then the proof is immediate. Suppose $N > 1$, let

$$\Theta = \frac{\text{Hess}^X(u)(g, h)}{-u \langle \nabla g, \nabla h \rangle}.$$

From Sturm's estimate, we get

$$2 \left[\Theta - \frac{1}{N} \right]^2 \leq \left(\frac{N-1}{N} \right) \left[\frac{N-2}{N} + 1 \right],$$

hence,

$$2 \left[\Theta - \frac{1}{N} \right]^2 \leq 2 \frac{(N-1)^2}{N^2},$$

which means

$$\frac{1}{N} - 1 \leq \Theta - \frac{1}{N} \leq 1 - \frac{1}{N},$$

or

$$-\frac{2}{N} - 1 \leq \Theta \leq 1.$$

Now, letting $N \rightarrow \infty$, one gets $|\Theta| \leq 1$. ■

5. $\frac{d^2}{2}$ -CONCAVITY OF f ON SUITABLE DOMAINS

In this section, we will prove that multiples of f (af for $a \in (-\frac{\pi}{2}, \frac{\pi}{2})$) are $c := \frac{d^2}{2} -$ concave on appropriate maximal domains. Roughly speaking, this is what we need in order to make sure that the gradient flow of f exists at least for short time. This will also imply that the gradient trajectories of f are geodesics at least for a short time. In the Riemannian setting this would mean that at least locally, the gradient of f is a parallel vector field.

Below, we will prove a series of lemmas that will lead to the proof of the $\frac{d^2}{2} -$ concavity of af for $a \in (\frac{\pi}{2}, \frac{\pi}{2})$ on some suitable maximal domains, \mathcal{D}_a .

Let

$$\begin{aligned} S_{-1} &:= \{x \in X : f(x) = -\frac{\pi}{2} \text{ (i.e. } u(x) = -1)\}, \\ &\text{and,} \\ S_{+1} &:= \{x \in X : f(x) = \frac{\pi}{2} \text{ (i.e. } u(x) = 1)\}. \end{aligned}$$

Obviously, $S_{\pm 1} \neq \emptyset$ are compact.

Definition 5.1 (horizontal geodesics). Any geodesic, θ joining a point $x_0 \in S_{-1}$ and a point $x_1 \in S_{+1}$ is called a horizontal geodesic.

Lemma 5.2. Any geodesic θ with $\text{Length}(\theta) = \pi$ joins a point $p_{-1} \in S_{-1}$ to a point $p_{+1} \in S_{+1}$.

Proof.

Suppose x and y are the end points of θ . Take γ_1 to be a geodesic joining x to S_{-1} and γ_2 to be a geodesic joining y to S_{+1} . Without loss of generality, we are also assuming that $\text{dist}(x, S_{-1}) \leq \text{dist}(y, S_{+1})$. We want to prove that there are points $p_{\pm 1}$ on θ for which we have $u(p_{\pm 1}) = \pm 1$ or equivalently $f(p_{\pm 1}) = \pm \frac{\pi}{2}$. Suppose not. This means that on θ we either have $-1 \leq u < 1 - \varepsilon$ or $-1 + \varepsilon < u \leq 1$. Again, using the gradient estimate comparison and knowing that $v = \sin(t)$, we can compute that $\text{Length}(\theta) < \pi$ which is a contradiction. ■

Lemma 5.3. Every $x \in X$ lies on a horizontal geodesic.

Proof.

Let γ_1 and γ_2 be geodesics joining x to S_{-1} and S_{+1} respectively (this can be assumed since $S_{\pm 1}$ are compact). We, in fact, need to prove that $\gamma := \gamma_1 + \gamma_2$ (i.e. the concatenation of γ_1 and γ_2) is also a geodesic. Using the fact that $|\nabla f| = 1$ and with computations similar to Lemma 2.17, one sees that $\text{Length}(\gamma_2) = \frac{\pi}{2} - f(x)$ and $\text{Length}(\gamma_1) = f(x) + \frac{\pi}{2}$. So $\text{Length}(\gamma) = \pi$ and γ is joining S_{-1} to S_{+1} , so γ has to be a geodesic. ■

Lemma 5.4.

$$f(p) = f(q) \iff \text{dist}(p, S_{-1}) = \text{dist}(q, S_{-1}) \quad \& \quad \text{dist}(p, S_{+1}) = \text{dist}(q, S_{+1}).$$

Proof.

Suppose γ_1 and γ_2 are the geodesic segments joining S_{-1} to p and p to S_{+1} respectively and θ_1 and θ_2 are the geodesic segments joining S_{-1} to q and q to S_{+1} respectively. Using the gradient comparison estimates and the fact that $|\nabla f| = 1$, one computes

$$\begin{aligned} \text{Length}(\gamma_1) &= f(p) + \frac{\pi}{2} = f(q) + \frac{\pi}{2} = \text{Length}(\theta_1), \\ &\quad \& \\ \text{Length}(\gamma_2) &= \frac{\pi}{2} - f(p) = \frac{\pi}{2} - f(q) = \text{Length}(\theta_2). \end{aligned}$$

■

Lemma 5.5. *The following are equivalent:*

- (1) $|f(x) - f(y)| = d(x, y)$,
- (2) $d(x, y) + \text{dist}(y, S_{+1}) + \text{dist}(x, S_{-1}) = \pi$ if $\text{dist}(x, S_{-1}) \leq \text{dist}(y, S_{-1})$,
- (3) x and y are on a single horizontal geodesic joining a point $p_{-1} \in S_{-1}$ to a point $p_{+1} \in S_{+1}$.

Proof.

(1) \rightarrow (2): It is easy to see that $d(x, y) + \text{dist}(y, S_{+1}) + \text{dist}(x, S_{-1}) \geq \pi$. Suppose $d(x, y) + \text{dist}(y, S_{+1}) + \text{dist}(x, S_{-1}) > \pi$. We also know that $\text{dist}(x, S_{-1}) = f(x) - (-\frac{\pi}{2}) = f(x) + \frac{\pi}{2}$ and $\text{dist}(y, S_{+1}) = \frac{\pi}{2} - f(y)$. So, we must have $f(x) - f(y) + d(x, y) > 0$ but this obviously contradicts $|\nabla f| = 1$.

(2) \rightarrow (3): Is obvious from Lemma 5.2.

(2) \rightarrow (3): Suppose θ is a horizontal geodesic containing both x and y . We have $\text{dist}(x, S_{-1}) = f(x) - (-\frac{\pi}{2}) = f(x) + \frac{\pi}{2}$ and $\text{dist}(y, S_{+1}) = \frac{\pi}{2} - f(y)$ and therefore, $d(x, y) = f(y) - f(x)$. Also from Lemma 6.4, we know $f(y) \geq f(x)$. Hence, we are done.

■

Lemma 5.6. *Fix $y \in X$. If $p \in X$ satisfies*

$$\inf_{x \in X} \frac{d(x, y)^2}{2} - f(x) = \frac{d(p, y)^2}{2} - f(p),$$

then, p and y are on the same horizontal geodesic.

Proof.

Suppose not. Pick a point q' on the horizontal geodesic containing y such that

$$f(q') = f(p).$$

So, we obviously have

$$d(q', y) + \text{dist}(y, S_{+1}) + \text{dist}(q', S_{-1}) = \pi.$$

Also we have

$$d(p, y) + \text{dist}(y, S_{+1}) + \text{dist}(p, S_{-1}) > \pi.$$

We want to prove that $d(q', y) < d(p, y)$. From above, we have

$$d(p, y) + \text{dist}(p, S_{-1}) > d(q', y) + \text{dist}(q', S_{-1}).$$

Also since $f(q') = f(p)$ (or equivalently, $u(q') = u(p)$), we get

$$d(q', S_{-1}) = d(p, S_{-1}) \quad \text{and} \quad d(q', S_{+1}) = d(p, S_{+1}),$$

hence, $d(q', y) < d(p, y)$. This means that q' is a better candidate for minimizing the quantity $\frac{d(x, y)^2}{2} - f(x)$ and this is a contradiction. ■

Lemma 5.7. Fix $y \in X$. For any x on the same horizontal geodesic as y , we have

$$f(x) = \begin{cases} f(y) + d(x, y) & : \text{dist}(x, S_{+1}) \leq \text{dist}(y, S_{+1}) \\ f(y) - d(x, y) & : \text{dist}(x, S_{-1}) \leq \text{dist}(y, S_{-1}), \end{cases}$$

and in particular,

$$d(x, y)^2 = (f(x) - f(y))^2.$$

Proof.

The proof is obvious from Lemma 5.5. ■

Now we are ready to compute the $c := \frac{d^2}{2}$ -Legendre transform of (af) ($a \in \mathbb{R}$).

Lemma 5.8. Let $a \in \mathbb{R}$, Then

$$(af)^c(y) = \begin{cases} -af(y) - \frac{a^2}{2} & \text{if } a \in [-\frac{\pi}{2} - f(y), \frac{\pi}{2} - f(y)] \\ \min \left\{ \frac{(\frac{\pi}{2} - f(y))^2 - a\pi}{2}, \frac{(\frac{\pi}{2} + f(y))^2 + a\pi}{2} \right\} & \text{if otherwise.} \end{cases}$$

Proof.

We need to minimize the following quantity

$$Q(t) := \frac{(t - f(y))^2}{2} - a(t),$$

over $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Taking the derivative with respect to t and equating to 0, we get

$$t = f(y) + a.$$

This means that the minimum is equal to $Q(a + f(y))$ if $a \in [-\frac{\pi}{2} - f(y), \frac{\pi}{2} - f(y)]$ and equal to $\min \{Q(-\frac{\pi}{2}), Q(\frac{\pi}{2})\}$ otherwise. ■

Lemma 5.9.

$$(af)^{cc}(y) = (af)(y) \quad \text{when,} \quad a \in [-\frac{\pi}{2} - f(y), \frac{\pi}{2} - f(y)].$$

Proof.

In this case, we compute

$$(af)^{cc}(y) = \left[-af(\cdot) - \frac{a^2}{2} \right]^c(y) = - \left(-af(\cdot) - \frac{a^2}{2} \right)(y) - \frac{a^2}{2} = af(y).$$

■

6. THE GRADIENT FLOW OF f AND ITS PROPERTIES

In this section, we will briefly highlight the key facts that will enable us to adapt Gigli's proof of the splitting theorem, for complete $RCD(0, N)$ spaces that contain a line (see Gigli [25, 26]), to our setting (compact space having a harmonic potential function, f that also satisfies $|\nabla f| = 1$ everywhere).

6.1. Short time existence. The following Theorem guarantees that the gradient flow of f exists for short time and is single valued.

Theorem 6.1 (existence and single value property). *Let $-\frac{\pi}{2} < a < \frac{\pi}{2}$,*

$$\mathcal{D}_a := \begin{cases} \mathcal{D}_{(-\frac{\pi}{2}, \frac{\pi}{2}-a)} & \text{if } a \geq 0 \\ \mathcal{D}_{(-\frac{\pi}{2}-a, \frac{\pi}{2})} & \text{if } a \leq 0 \end{cases},$$

where, $\mathcal{D}_{c,d} := f^{-1}[(c, d)]$; and

$$\mathcal{R}_a := \begin{cases} \mathcal{D}_{(-\frac{\pi}{2}+a, \frac{\pi}{2})} & \text{if } a \geq 0 \\ \mathcal{D}_{(-\frac{\pi}{2}, \frac{\pi}{2}+a)} & \text{if } a \leq 0 \end{cases}.$$

Then, there exists a unique Borel map $\Psi_a(\cdot) : \mathcal{D}_a \rightarrow \mathcal{P}(X)$ for which the following hold

- (E₁) $\text{supp}[\Psi_a(x)] \subset \partial^c((a)f)(x)$ for m -a.e. $x \in \mathcal{D}_a$.
- (E₂) $(\Psi_a)_\#(m|_{\mathcal{D}_a}) \ll m$
- (E₃) $\Psi_a(x) = \delta_{\Psi_a(x)}$ for m -a.e. $x \in \mathcal{D}_a$ where, $\Psi_a : X \rightarrow X$ is an invertible Borel map that satisfies $[\Psi_a]_\#(m|_{\mathcal{D}_a}) = m|_{\mathcal{R}_a}$.

Proof.

Without loss of generality, assume $N > 1$ and $0 < a < \frac{\pi}{2} - \delta$. Fix a Borel subset $E \subset \mathcal{D}_a$ and let $E' := T_a(E) \cap f^{-1}((a, \frac{\pi}{2}])$ ($T_a(E)$ is the a -tubular neighbourhood of E).

Now for fixed $\varepsilon < \delta$, consider the map $x \mapsto \partial^c((a+\varepsilon)f)(x)$ from \mathcal{D}_a to closed subsets of X . By the Kuratowski and Ryll-Nardzewski's *Borel Selection* Theorem (see Kuratowski-Ryll-Nardzewski [37] and [25, Theorem 3.1]), there exists a map $\Psi_{a,\varepsilon} : \mathcal{D}_a \rightarrow X$ with $\Psi_{a,\varepsilon}(x) \in \partial^c((a+\varepsilon)f)(x)$ for all $x \in \mathcal{D}_a$.

Let $\mu := m(E)^{-1}m|_E$ be the uniform probability measure on E and let $\nu := [\Psi_{a,\varepsilon}]_\# \mu$. Take $\pi \in \text{OptGeo}(\mu, \nu)$ and let π_x be the disintegration with respect to the evaluation map at time 0, (i.e. e_0). Since $(a+\varepsilon)f$ is a Kantorovitch potential on \mathcal{D}_a , we have $\text{supp}([e_1]_\# \pi_x) \subset \partial^c((a+\varepsilon)f)$. Let $\sigma := \left[e_{\frac{a}{a+\varepsilon}} \right]_\# \pi = \rho m + \sigma_{\text{sing}}$. Using the Jensen's inequality one gets

$$\mathcal{U}_N(\sigma) \leq -\frac{a}{a+\varepsilon} m(E)^{\frac{1}{N}}.$$

Also, it is easy to see that σ is concentrated on E' , hence, with a similar computation as in [25, Theorem 3.18], we get

$$\int_X \rho \, dm \geq \left(\frac{m(E)}{m(E')} \right)^{\frac{1}{N-1}} \left(\frac{a+\varepsilon}{a} \right)^{-\frac{N}{N-1}}.$$

Again similar to [25, Theorem 3.18], we choose the Borel set, A that satisfies

$$(1) \text{ supp}(\rho m) \subset A, \quad (2) \sigma_{\text{sing}}(A) = 0, \quad (3) \left[e_{\frac{a}{a+\varepsilon}} \right]_{\#} \pi|_{e_{\frac{a}{a+\varepsilon}}^{-1}(A)} = \rho m.$$

Now, obviously we have

$$[e_0]_{\#} \pi|_{e_{\frac{a}{a+\varepsilon}}^{-1}(A)} = \rho_0 m \ll \mu (\text{and } \ll m).$$

Let $E_0 := \rho_0 > 0$ and compute

$$m(E_0) \geq \frac{m(E)^{\frac{N}{N-1}}}{m(E')^{\frac{1}{N-1}}} \left(\frac{a+\varepsilon}{a} \right)^{-\frac{N}{N-1}}.$$

This means that E_0 takes up a definite portion of E .

Now, on E_0 , we define the measure valued map, Ψ_a as follows

$$\Psi_a(x) := \left(\left(e_0, e_{\frac{a}{a+\varepsilon}} \right)_{\#} \pi|_{e_{\frac{a}{a+\varepsilon}}^{-1}(A)} \right)_x,$$

where, the subscript, x means the disintegration with respect the map $\text{proj}_1 : X \times X \rightarrow X$. Now replace E by $E \setminus E_0$ and repeat the above construction to get a subset $E_1 \subset E \setminus E_0$ with the following estimate on its measure

$$m(E_1) \geq \frac{m(E \setminus E_0)^{\frac{N}{N-1}}}{m(E')^{\frac{1}{N-1}}} \left(\frac{a+\varepsilon}{a} \right)^{-\frac{N}{N-1}}.$$

Now the proof of (E_1) and (E_2) can be completed by iterating this construction (see [25, Theorem 3.18] for a more detailed proof). The proof of **single value property** (i.e. (E_3)) can be carried out verbatim as in [25, Proposition 3.20], so it will be omitted. ■

6.2. Short time uniqueness and measure preserving property. Using Theorem 6.1 and following the ideas of Gigli [26, Theorem 3.21], we can now, prove the following theorem.

Definition 6.2 (effective domain). The set \mathcal{A} defined by

$$\mathcal{A} := \left\{ (x, t) \in X \times \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) : |f(x) + t| < \frac{\pi}{2} \right\},$$

is called the *effective domain* of the gradient flow of f .

Theorem 6.3 (uniqueness and measure preserving property). *Let \mathcal{A} be the effective domain. There exists a Borel map $\Psi : \mathcal{A} \subset X \times \mathbb{R} \rightarrow X$ that satisfies the following*

- (GF₁) *The map $t \mapsto \Psi_t(\cdot)$ is a gradient flow trajectory of f for a.e. $t \in \text{proj}_2(\mathcal{A})$ and $\Psi_0 = \text{Id}$.*
- (GF₂) *$[\Psi_t]_{\#} m|_{\mathcal{D}_t} = m|_{\mathcal{R}_t}$ for all $t \in \text{proj}_2(\mathcal{A})$.*
- (GF₃) *For any $t, s \in \mathbb{R}$, it holds $\Psi_t \circ \Psi_s(x) = \Psi_{t+s}(x)$ for m -a.e. $x \in \mathcal{D}_{t+s}$. (This is characterizes the gradient flow of f as a groupoid action on the underlying space)*
- (GF₄) (uniqueness). *If $\mathbf{F} : X \rightarrow \mathcal{P}(C([t_0, t_1], X))$ is a Borel map that satisfies*
 - (U₁) *γ is a gradient flow trajectory of f with $\gamma(0) = x$ for $\mathbf{F}(x)$ -a.e. γ .*
 - (U₂) *For any $t_0 \leq t \leq t_1$, $[\mathbf{F}_t]_{\#} m \ll m$ where $\mathbf{F}_t : X \rightarrow \mathcal{P}(X)$ is defined by $\mathbf{F}_t(x) := [e_t]_{\#}(\mathbf{F}(x))$.**Then, $\mathbf{F}(x)$ is concentrated on the curve $[t_0, t_1] \ni t \mapsto \mathbf{F}_t(x)$ for m -a.e. $x \in X$.*

Proof Sketch.

The proof of the **Measure Preservation Property** relies on an *energy inequality estimate* as is proven in [25, Proposition 3.19]. The main ingredient of the proof is the weak harmonicity of the

potential function ($\Delta f = 0$ in our case). The rest of the proof can be repeated verbatim and hence we will refer the reader to the detailed proof of Propositions 3.19 and 3.20 and Theorem 3.21 in Gigli [25]. ■

6.3. Short time distance preserving property. As we observed in the last section, the gradient flow of f is defined on appropriate domains. In a nutshell, from last section we know that, if $0 \leq a < \pi$, then Ψ_a is defined on $\mathcal{D}_a := f^{-1} \left[\left(-\frac{\pi}{2}, \frac{\pi}{2} - a \right) \right]$. In order to prove that the gradient flow of f , when defined, preserves the distance, we will proceed as in Gigli [25] i.e. we will prove that it preserves the Dirichlet energy. To make the proof rigorous we will proceed as follows:

6.3.1. $\mathcal{D}_{c,d} := f^{-1}[(c,d)]$ is geodesically convex.

Lemma 6.4. *Suppose γ is a geodesic in X , then $f \circ \gamma(t)$ is monotone.*

Proof.

Obviously, we have $\text{Length}(\gamma) \leq \pi$. Since $|\nabla f| = 1$, we have $|f(\gamma_{t_1}) - f(\gamma_{t_0})| = t_1 - t_0$. Suppose $f \circ \gamma$ has an extrema at $t' \in (t_0, t_1)$. Then it is easy to compute that $|f \circ \gamma|(t') = 0$. The chain rule then will imply that $|\nabla f|(\gamma(t')) = 0$ which is a contradiction. ■

Lemma 6.5. *Consider the set $\mathcal{D}_{a,b} := f^{-1}[(a,b)]$. Then $\mathcal{D}_{a,b}$ is a geodesically convex subset of X .*

Proof.

Take two points $x, y \in \overline{\mathcal{D}_{a,b}} = f^{-1}([a,b])$ and a geodesic γ between them. From the previous lemma, we know that f is monotone along γ . Pick $p = \gamma(\bar{t}) \in \text{Im}(\gamma)$ and $q \in \mathcal{D}_{a,b}$ such that

$$d(p, q) = \text{dist}(p, \mathcal{D}_{a,b}) = \sup_{x \in \text{Im}(\gamma)} \text{dist}(x, \mathcal{D}_{a,b}),$$

and without loss of generality suppose $f(p) > b$.

Claim For every $t \in (t_0, t_1)$, we have $f(\gamma(t)) \leq f(p)$. Suppose not. Let $f(\gamma(t)) > f(p)$ and without loss of generality assume $t > \bar{t}$. We also know that $d(\gamma(t), q) \leq d(p, q)$. Let θ be a geodesic joining q to $\gamma(t)$. We know that f is monotone along θ and also

$$\text{Length}(\theta) = f(\gamma(t)) - f(q) > f(p) - f(q) = d(p, q),$$

and this is a contradiction.

Hence, we have $f \circ \gamma$ attains its maximum at $p \in \text{Im}(\gamma)$ which contradicts the monotonicity of $f \circ \gamma$. Now, since

$$\mathcal{D}_{a,b} = \bigcup_n \overline{\mathcal{D}_{a+\frac{1}{n}, b-\frac{1}{n}}},$$

we can deduce that, $\mathcal{D}_{a,b}$ is geodesically convex (since the direct limit of nested geodesically convex sets is geodesically convex). ■

Now since $\mathcal{D}_{a,b}$ is geodesically convex, we have

$$(D_{a,b}, d|_{D_{a,b}}, m),$$

is an $RCD(0, N)$ space.

6.3.2. *Short time invariance under the heat flow.* Our goal in this section is to prove that f is invariant under the heat flow of X .

Theorem 6.6 (invariance under heat flow). *Suppose f is as before, then one has*

$$(I_1) \quad \mathbf{H}_t(f) = f(x) \quad \forall x \in X,$$

(I₂) and furthermore, we have the point-wise Euler's equation:

$$\mathbf{H}_t \langle \nabla f, \nabla g \rangle = \langle \nabla f, \mathbf{H}_t(g) \rangle.$$

Proof.

Let $\rho(t, x, y)$ be the heat kernel. Then, since $\int_X \rho(t, x, y) \, dm(y) = 1$ is true for any t , we will get

$$f(x) = \int_X f(y) \rho(t, x, y) \, dm(y).$$

Now, since, $|\nabla f| = 1$ (in particular, f is 1-Lipschitz), we can compute

$$\begin{aligned} |\mathbf{H}_t(f)(x) - f(x)| &= \left| \int_X f(y) \rho(t, x, y) \, dm(y) - \int_X f(x) \rho(t, x, y) \, dm(y) \right| \\ &\leq \int_X d(x, y) \rho(t, x, y) \, dm. \end{aligned}$$

Using the Gaussian estimates for the heat kernel (5), one sees that

$$\int_X d(x, y) \rho(t, x, y) \, dm \leq Ct^{\frac{1}{2}},$$

which on letting $t \rightarrow 0$, will give us

$$\lim_{t \rightarrow 0} \mathbf{H}_t(f)(x) = f(x) \quad \forall x \in X.$$

So, to complete the proof, we need to argue that for positive times $t > 0$, the rate of change of $\mathbf{H}_t(f)(x)$ is zero. Fix $x_0 \in X$ then for $t > 0$, one gets

$$\begin{aligned} \frac{d}{dt} \mathbf{H}_t(f)(x) &= \frac{d}{dt} \int_X f(y) \rho(t, x, y) \, dm(y) \\ &= \int_X f(y) \frac{d}{dt} \rho(t, x, y) \, dm(y) \\ &= \int_X f(y) \Delta_{\text{Ch}} \rho(t, x, y) \, dm(y) \\ &= \int_X [\Delta_{\text{Ch}} f(y)] \rho(t, x, y) \, dm(y) \\ &= 0. \end{aligned}$$

For more details, see Gigli [25]. ■

To prove (I₂), one can apply the Bakry-Émery contraction, (6), to $f + \varepsilon g$ to get

$$\begin{aligned} 2\varepsilon \langle \nabla f, \nabla \mathbf{H}_t(g) \rangle + \varepsilon^2 |\nabla \mathbf{H}_t(g)|^2 &= |\nabla \mathbf{H}_t(f + \varepsilon g)|^2 - 1 \\ &\leq \mathbf{H}_t |\nabla(f + \varepsilon g)|^2 - 1 \\ &= 2\varepsilon \mathbf{H}_t \langle \nabla f, \nabla g \rangle + \varepsilon^2 \mathbf{H}_t |\nabla g|^2; \end{aligned}$$

dividing by ε and letting $\varepsilon \rightarrow 0$, gives the desired result.

6.3.3. $(\cdot) \circ \Psi_t$ is an isometry between $W^{1,2}$ spaces for short time. In this section, we want to prove that right composition with the flow Ψ_t is an isometry from the Sobolev space, $W^{1,2}(\mathcal{R}_a)$ into the Sobolev space, $W^{1,2}(\mathcal{D}_a)$ at least for short time (i.e. $a \in (-\frac{\pi}{2}, \frac{\pi}{2})$).

Theorem 6.7. *Let $g \in W^{1,2}(\mathcal{R}_a)$ and let $|a| < \frac{\pi}{2}$, then $g \circ \Psi_a \in W^{1,2}(\mathcal{D}_a)$ and moreover, $\|g\|_{W^{1,2}(\mathcal{R}_a)} = \|g \circ \Psi_a\|_{W^{1,2}(\mathcal{D}_a)}$.*

Proof Sketch.

The proof is quite similar to the proof of [25, Proposition 4.16]. We only highlight the adjustments one needs to make.

From Theorem 6.3, we know that Ψ_a is measure preserving when $|a| < \frac{\pi}{2}$ and from \mathcal{D}_a to \mathcal{R}_a . Also recall that the set

$$G := \left\{ \Psi_a : |a| < \frac{\pi}{2} \right\},$$

has a *groupoid* structure (i.e every element has an inverse). It is not closed under addition, but for every $|a| < \frac{\pi}{2}$, and $|t| < \frac{\pi}{2} - |a|$, one has

$$\Psi_a \circ \Psi_{\pm t} = \Psi_{a \pm t}, \quad \text{on } \mathcal{D}_{t+s}.$$

Then, for $|s - t|$ small enough, we can compute

$$\begin{aligned} \int_X |g \circ \Psi_s - g \circ \Psi_t| \, dm &= \int_X |g \circ \Psi_{s-t} - g| \, dm \\ &\leq |s - t| \int_X |\nabla g|^2 \, dm = |s - t| \|g\|_{L^2(\mathcal{R}_s)}. \end{aligned}$$

This means that right composition with Ψ_a for short time is Lipschitz and with Lipschitz constant $= \|g\|_{L^2(\mathcal{R}_s)}$.

The rest of the proof is just the same as in [25, Proposition 4.16] and hence, omitted here. ■

From Theorem 6.7, it follows that at least for short time, the gradient flow, Ψ_a , preserves the Dirichlet energy of functions, g with $\text{supp}(g) \subset \mathcal{D}_a$. Hence, the following holds.

Corollary 6.8. *Let $-\frac{\pi}{2} < a < \frac{\pi}{2}$. Consider the map $\Psi_a : \mathcal{D}_a \rightarrow \mathcal{R}_a$ between $RCD(0, N)$ spaces. Then, Ψ_a preserves the Dirichlet energy.*

Proof.

Straightforward from Theorem 6.7. ■

Theorem 6.9. $\Psi_a : \mathcal{D}_a \rightarrow \mathcal{R}_a$ is distance preserving.

Proof.

The proof is a direct application of Lemmas 4.17 and 4.19, Proposition 4.20 and Theorem 4.25 from Gigli [25]. ■

Remark 6.1. *By now, it should be more or less clear how the gradient flow of f looks like. for any x , the gradient flow trajectory of f starting from x is the geodesic joining x to the extremal set $S_{+1} := f^{-1}(\frac{\pi}{2})$. This means that, the smaller $f(x)$ is, the longer the life-time of the gradient flow trajectory starting from x will be.*

7. THE QUOTIENT SPACE AND SPLITTING PHENOMENON

In this section, we will define the quotient space, X' , just as in Gigli [25] and will show that it isometrically embeds into X .

7.1. The quotient metric space.

Definition 7.1 (the quotient space). Consider the following relation

$$x \sim y \iff \Psi_t(x) = y \text{ for some } t \in \mathbb{R}.$$

It is easy to see that \sim is an equivalence relation on X . The quotient space X' , then is taken to be $X' = \frac{X}{\sim}$. The natural projection is denoted by $\mathbf{P} : X \rightarrow X'$. The distance d' on X' is defined by

$$d'(\mathbf{P}(x), \mathbf{P}(y)) := \inf \left\{ d(\Psi_t(x), y) : -\frac{\pi}{2} - f(x) < t < \frac{\pi}{2} - f(x) \right\}.$$

The right inverse of \mathbf{P} is denoted by $\iota : X' \rightarrow X$ and is given by: $\iota(x') = x$ if $\mathbf{P}(x) = x'$ and $f(x) = 0$.

Remark 7.1. It is easy to see that X' is isometric to $Z := f^{-1}(\{0\})$.

Now we can prove that the X' isometrically embeds into $\hat{X} := X \setminus (S_{-1} \sqcup S_{+1})$.

Theorem 7.2. X' isometrically embeds into \hat{X} .

Proof.

In order to prove that X' isometrically embeds into \hat{X} , it suffices to prove that for two points $x' = \mathbf{P}(x)$ and $y' = \mathbf{P}(y)$, the minimum of $d(x, \Psi_t(y))$ is achieved at t_0 where, $f(x) = f(\Psi_{t_0}(y))$. This fact follows from the proof of Lemma 5. \blacksquare

7.1.1. The quotient measure, m' .

Definition 7.3. The measure m' on X' is defined as follows

$$m'(A) := m(\mathbf{P}^{-1}(A) \cap f^{-1}([0, 1])) \text{ for all Borel sets } A \subset X'.$$

7.2. The space splits. So far, we know that the set,

$$\hat{X} := X \setminus (S_{-1} \sqcup S_{+1}) = \mathcal{D}_{(-\frac{\pi}{2}, \frac{\pi}{2})}.$$

is a complete $RCD(0, N)$ space by Lemma 6.5. The following theorem can be proven by combining the results of Sections 3.4 and 2.5 of Gigli [26] and so, we are not going to repeat the arguments.

Define the maps $\mathsf{T} : X' \times (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \hat{X}$ and $\mathsf{S} : \hat{X} \rightarrow X' \times (-\frac{\pi}{2}, \frac{\pi}{2})$ as follows:

$$\mathsf{T}(x', t) := \Psi_{-t}(\iota(x')) \quad \text{and} \quad \mathsf{S}(x) := (P(x), f(x)).$$

Theorem 7.4. The following hold.

(S₁) T and S are homeomorphisms.

(S₂) Suppose $g \in S_{loc}^2(\mathcal{D}_a)$. For any $x' \in X'$ and $-\frac{\pi}{2} < t < \frac{\pi}{2} - a$ (if $a > 0$) and $-\frac{\pi}{2} - a < t < \frac{\pi}{2}$ (if $a < 0$), define $g^{(x')}(t) := g(\Psi_t(\iota(x')))$; then, for $m' - a.e. x' \in X'$, it holds $g^{(x')} \in S_{loc}^2(\mathcal{D}_{a-t})$ and

$$\left| \nabla g^{(x')} \right| (t) \leq |\nabla g| [\Psi_t(\iota(x'))] \quad m' \otimes \mathcal{L}^1|_{(-\frac{\pi}{2} < t < \frac{\pi}{2})}(x', t) \in X' \times \left(-\frac{\pi}{2} < t < \frac{\pi}{2} \right).$$

(S₃) Suppose $h \in S_{loc}^2\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$. Let $\bar{h} := h \circ f : X \rightarrow \mathbb{R}$. Then,

$$|\nabla \bar{h}|(x) = |\nabla h|[f(x)] \quad m - a.e. x \in X.$$

(S₄) Pythagoras Theorem holds, namely, $d(\mathsf{T}((x', t)), \mathsf{T}((y', s))) = [d'(x', y')^2 + |t - s|^2]^{\frac{1}{2}}$.

Theorem 7.4 will then imply that right composition with S and left compositions with T preserves the Euler energy (when restricted to a dense sub-algebra of the Sobolev space, $W^{1,2}(\hat{X})$ and $W^{1,2}(X' \times (-\frac{\pi}{2}, \frac{\pi}{2}))$ respectively. This means that the maps S and T preserve the Dirichlet energies of all Sobolev functions which in turn, implies that these maps are distance preserving. Using a, now standard, dimension reduction argument as in Cavaletti-Sturm [14], one gets

Theorem 7.5. (X', d', m') is an $RCD(0, N - 1)$ space.

8. X IS ONE DIMENSIONAL

Here we will complete the proof of Theorem 1.1. We will argue that, in the virtue of our splitting result, and the fact that $\text{diam } X = \text{Length}(\theta) = \pi$ for any maximal horizontal geodesic, θ , one must have Hasdorff dimension, $\mathcal{H}_{dim}(\hat{X}) = 1$ (otherwise, Pythagorea's theorem would be violated).

Lemma 8.1. Suppose C is a connected component of X' , then C is a single point space

Proof.

Suppose not. Take two distinct points $x', y' \in C$ with $d(x', y') > \delta > 0$.

Now, choose an $0 < s < \frac{\pi - \sqrt{\pi^2 - \delta^2}}{2}$ and take two points $x := \mathsf{T}\left(-\frac{\pi}{2} + s, x'\right)$ and $y := \mathsf{T}\left(\frac{\pi}{2} - s, y'\right)$ in X . Using the Pythagorea's theorem for isometric products, we can compute

$$d(x, y) = \sqrt{(\pi - 2s)^2 + \delta^2} > \pi = \text{diam}(X).$$

This is obviously a contradiction. ■

We also need the following elementary lemma that characterizes the connected components of $\hat{X} := X \setminus (S_{-1} \sqcup S_{+1})$.

Lemma 8.2. Let \hat{C} be a connected component of \hat{X} , then, there exists a point $x' \in X'$ such that

$$\hat{C} = \mathsf{T}\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \{x'\}\right).$$

Proof.

Take a point $x \in \hat{C}$ and let $x' := \mathbf{P}(x) \in X'$. Obviously we have,

$$\mathsf{T}\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \{x'\}\right) \subset \hat{C}.$$

Now suppose there exists a point $y \in \hat{C} \setminus \mathsf{T}\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \{x'\}\right)$. This means that $\mathbf{P}(y) = y' \neq x'$ i.e. x' and y' are two distinct connected components in X' . Let γ be a geodesic in \hat{X} joining x and y . Then $\text{proj}_2 \circ \mathsf{S}(\gamma)$ is a curve in X' joining x' and y' and this is a contradiction. ■

Now we know that \hat{X} is comprised of disjoint open line segments of length π . The following lemma asserts that, we can at most have two of these open line segments.

Lemma 8.3. *\hat{X} has at most two connected components.*

Remark 8.1. *In the smooth setting, as we saw in Lemma 3.2, one has*

$$\text{Hess } u = -u \nabla(\cdot) u \otimes \nabla(\cdot),$$

and with a little work (using the exponential map as in Hang-Wang [31, Lemma 2]), one sees that for any $p_{+1} \in S_{+1}$, there corresponds at most two points $p_{-1} \in S_{-1}$ and vice versa. Therefore, $2 \leq \#(S_{+1} \sqcup S_{-1}) \leq 4$. In the non-smooth setting, even though, there is a notion of exponentiation (see Gigli-Rajala-Sturm [29]), as we discussed in Remark 4.1, we could not quite (prior to proving the splitting phenomenon) prove the Hessian identity $\text{Hess}^X(u)(f, g) = -u \langle \nabla f, \nabla g \rangle$. Even if, we take the Hessian identity for granted, it is still not clear to us how we can use it along with the exponentiation to assert that $2 \leq \#(S_{+1} \sqcup S_{-1}) \leq 4$.

Before, we get to the proof of Lemma 8.3, we need the following lemma that asserts that the letter "Y" space (also called the *tripod*) fails to be *essentially non-branching* and in particular, is not an RCD space.

Lemma 8.4. *Let X be the wedge of three segments of length 1 (the letter "Y" space) equipped with its Lebesgue measure. Then X is not essentially non-branching.*

Proof.

One can directly resort to the characterization theorem of Kitabeppu and the author (see Kitabeppu-Lakzian [35]) to see that the letter Y space is not RCD . But, we will also provide an elementary proof here. We will prove that X is not essentially non-branching. For that, it suffices to find two absolutely continuous measures $\mu, \nu \in \mathcal{P}(X)$ and an optimal dynamical plan $\pi \in \mathbf{OptGeo}(X)$ that is concentrated only on branching geodesics.

The construction is actually very intuitive. Let x, y, z denote the mid points of each branch. Let

$$\mu := \delta_y \quad \& \quad \nu := \frac{1}{2} \left(\mathcal{L}(B_\delta(x))^{-1} \mathcal{L}|_{B_\delta(x)} + \mathcal{L}(B_\delta(z))^{-1} \mathcal{L}|_{B_\delta(z)} \right).$$

for some $\delta < \frac{1}{4}$. Let π_1 be an optimal coupling between ν and $\mathcal{L}(B_\delta(x))^{-1} \mathcal{L}|_{B_\delta(x)}$ and π_2 , an optimal coupling between ν and $\mathcal{L}(B_\delta(z))^{-1} \mathcal{L}|_{B_\delta(z)}$. Then it is easy to see that $\frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$ is an optimal coupling between ν and μ , that is supported only on branching geodesics. This means that letter Y is not an strongly $CD(K, N)$ space and in particular, it is not an $RCD(K, N)$ space. ■

Proof of Lemma 8.3.

Without loss of generality, suppose \hat{X} consists of three connected components (open intervals of length π as we saw in Lemma 8.2). Let $f^{-1}(\frac{\pi}{4}) = \{x, y, z\}$. Now adding the sets $S_{\pm 1}$ back to \hat{X} , and repeating a similar argument as in Lemma 8.4, we see that, the space X fails to be essentially non-branching and therefore, is not a strongly $CD(K, N)$ space and in particular, it is not an $RCD(K, N)$ space. This is a contradiction. ■

The proof of Theorem 1.1 is almost complete, we just need to notice that any point in $S_{\pm 1}$ is a limit of points in \hat{X} , and \hat{X} has at most two components so $2 \leq \#(S_{-1} \sqcup S_{+1}) < 4$. Since $\text{diam } X = \pi$, we conclude that both S_{-1} and S_{+1} are single points (notice that $\#(S_{-1} \sqcup S_{+1}) = 3$ would imply that $\text{diam}(X) = 2\pi$). So X is either a circle of perimeter 2π or a closed interval of length, π .

Remark 8.2 (final remark). *Using similar arguments as in Kitabeppu-Lakzian [35], one sees that, when X is a circle, the measure has to be in the form of $c\mathcal{H}_d$ (for some positive constant, c) and If X is a line segment then, the measure, m , is of the form $e^{-W}\mathcal{H}_d$ for some $(0, N)$ -convex weight function, W .*

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[†] SAJJAD LAKZIAN, HAUSDORFF CENTER FOR MATHEMATICS, D-53115, BONN, GERMANY
 URL: <http://wt.iam.uni-bonn.de/sajjad-lakzian>
 E-mail address: lakzians@gmail.com